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Constraining Neutron Star Nuclear Equations of State Based on Observational Data

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for graduation with Honors

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Contents

1 The Death of a Star

1.1 Overview

Stars form when a dust cloud begins to collapse on itself. When the cloud begins to shrink, it starts to heat up. At a certain point, it becomes hot enough that nuclear fusion begins. This stops the collapse of the cloud and a star is born. In a star, nuclear fusion occurs when two nuclei, usually hydrogen, fuse together to become helium. In larger stars, heavier elements will also fuse together as well. This reaction releases large amounts of energy, which is emitted from the star as light and neutrinos (which are very small particles that are very hard to detect but normally are released in nuclear reactions). This energy is what counteracts the initial collapse of the cloud. This reaction lasts for millions to billions of years in a star (depending on the size of the star) until the star runs out of fuel for this reaction. At this point, the star begins to collapse because there is nothing to counteract the gravitational pull all the atoms in the star.

At some point, another pressure may stop this collapse. With the exception of very large stars, there will be a degeneracy pressure to stop this. In small stars, electron degeneracy pressure will stop the collapse and the resulting body is called a white dwarf. In stars with larger masses, they will collapse into neutron stars due to neutron degeneracy pressure. They are called neutron stars, as they are made up of almost entirely of neutrons. Stars that can overcome this pressure keep collapsing and become a singularity. This is a point that is of infinite density and is not well understood. The singularity and the event horizon, which is the area around the singularity where the escape speed is the speed of light is called a black hole [1].

This degeneracy pressure that resists the collapse is a quantum mechanical effect. The Pauli Exclusion Principle states that no two fermions can ever be in the same quantum state [2]. In stars that are collapsing, all the particles in the star are pushed very close to each other. Because of Pauli Exclusion, there is a pressure exerting back on the collapse of the star because the particles cannot occupy the same quantum state. As a way to visualize this process, we can look at the several quantum mechanical principles.

Every particle can be represented by some wave function that satisfies Schrödinger's equation. Below is Schrödinger's equation expressed in one dimension

$$i\hbar\frac{\partial^2\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{1.1}$$

where V is potential energy, Ψ is the wave function, $\hbar$ is Planck's constant, m is the mass of the particle, t is time, x is position [2]. The square of this wave function gives the probability density for the measurement of the position of the particle. The energy of a particle in a one dimensional box is as follows,

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{1.2}$$

where L is the length of the box, m is the mass of the particles, $n = 1, 2, 3\ldots,$

and $\hbar$ is Planck's constant. In three dimensions this would be

$$E_n = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \tag{1.3}$$

where each of the quantum numbers n_x, n_y, n_z can be $1, 2, 3, \ldots$. It can then be shown by the Heisenberg uncertainty principle,

$$\Delta p \Delta x \geq \frac{\hbar}{2}, \tag{1.4}$$

that with a small decrease in x or the wavelength of this wavefunction, the momentum, p has to increase to preserve this principle. So this is the mechanism providing the degeneracy pressure.

However, we expect that there is a maximum mass beyond which the degeneracy pressure cannot halt the collapse.This argument was given originally by L. Landau [3] and leads to a very rough estimate of the critical mass of a white dwarf. We imagine the star is made up of N nucleons and N electrons with a radius R. The protons contribute almost all of the mass of the star so the gravitational potential energy is approximately

$$E_g \sim -\frac{G(Nm_N)^2}{R}, \tag{1.5}$$

where m_N is the mass of a nucleon. The electrons spread throughout the star because the electrons spread out to match the distribution of the positively charged protons so the volume is approximately R^3. So the volume occupied by each electron is R^3/N. This volume should be roughly

$$V \sim \lambda^3 \tag{1.6}$$

where λ is the typical de Broglie wavelength. With some rearrangement,

$$\lambda \sim R/N^{1/3}. \tag{1.7}$$

Their typical momentum is then

$$p = \frac{h}{\lambda} \sim \frac{hN^{1/3}}{R}. \tag{1.8}$$

If we assume the electrons are highly relativistic, so that each electron has energy $E = pc$, the total electron energy is

$$E_e \sim Npc = \frac{hcN^{4/3}}{R}, \tag{1.9}$$

The total energy of the star is then the gravitational energy plus the energy of the electrons:

$$E \sim -\frac{G(m_N N)^2}{R} + \frac{hcN^{4/3}}{R}. \tag{1.10}$$

From this one can see the essential point: as N increases, the first term must eventually begin to dominate, making the energy negative. This means the star finds it energetically favorable to collapse to a smaller radius. The critical value of N will occur when the two terms are equal in magnitude, which gives

$$N_{max} \sim \left(\frac{hc}{Gm_N^2}\right)^{3/2}. \tag{1.11}$$

Since there is the same number of protons and electrons in our model, the maximum mass is

$$M_{max} = N_{max} m_N \sim 10 M_{\odot}. \tag{1.12}$$

Beyond this, the energy cost of shrinking arising from quantum mechanics is outweighed by the decrease in the gravitational energy of the system. It will continue to shrink unless some other effect intervenes. For a while dwarf, this upper limit was first calculated precisely to be $M_{max} = 1.4 M_{\odot}$ by Chandrasekhar [4], and is known as the Chandrasekhar limit. We will reproduce this famous calculation below.

Next, we will describe what happens to the star if the electron degeneracy pressure fails to prevent the collapse.In this case, the electrons and protons are pushed so close together that they undergo inverse beta decay, producing neutrons and neutrinos:

$$p + e^- \to n + \bar{\nu} \tag{1.13}$$

The neutrino has very little mass and escapes from the star. The result is a star made almost entirely out of neutrons, hence the name "neutron star." In this case, the neutron degeneracy pressure and the strong nuclear forces between the neutrons can perhaps stabilize the star. But if the star has enough mass to overcome these pressures, it will continue to collapse. At present we know of no other mechanism that can halt this collapse; the star shrinks to a very small size and becomes a black hole.

The presence of strong interactions between the neutrons makes estimating the upper mass limit for neutron stars much more difficult than for white dwarfs, where the electrons (as we will discuss) are essentially non-interacting.

In practice one needs the "equation of state" (EoS) for the neutrons making up the star, which is the relation between the energy density and the pressure, but this is not well known.It is difficult to make reliable calculations based on the strong interaction except in certain energy regimes. At high energies, for example in high-energy scattering processes, asymptotic freedom [5, 6] insures that ordinary perturbation theory is reliable. In addition, Chiral Perturbation Theory [7] is a useful technique at low energies. Outside of these regimes, however, strong interaction calculations are very difficult. The essential problem is that strongly interacting systems are always many-body systems, since long wavelength gluons can be produced immensely. Even the basic structure of a single nucleon is a very complex many-body problem that would be difficult to model.

Traditional nuclear physics is also of limited help. Here we have various semi-empirical formulas that capture a great deal of nuclear structure and properties, but the nuclear states we experience in our laboratories are roughly "symmetric," that is, containing roughly equal numbers of protons and neutrons. The matter in a neutron star is essentially entirely neutrons. In addition, the density of a neutron star is roughly 10 times larger than in a typical nucleus. We have very few clues regarding the behavior of such an exotic form of matter. In practice, one must resort to models for the nuclear matter equation of state, based on various theoretical ideas. One reason for studying neutron stars is that observations of the stars can help constrain the various models.

1.2 Goals and Outline

The general goal of this project is to study the effects of different model equations of state on the critical mass of a neutron star. This is an active area of research, and recently a new set of EoS's has been proposed [8], which aims to be "minimally constrained," that is, to be as general as possible. Our main goal is to calculate neutron star mass limits based on these, and and see how the observational data can constrain them.

To layout our plan, we will first discuss the white dwarf and reproduce the Chandrasekhar limit. This will provide us with a degenerate fermion EoS as well as an illustration of the methods used to determine the equilibrium states of a star. This involves the formulation of a set of coupled differential equations expressing the equilibrium conditions, which will need to be solved numerically.

Then we will attempt to use the same strategy with neutron stars by adding corrections due to special and general relativity, and employing a degenerate neutron EoS. This allows us to reproduce a limit first obtained by Oppenheimer and Volkoff [9], which, however, is not realistic since it neglects the strong interactions between neutrons. After these warmup problems, where the techniques for simulating dense stars are developed and understood, we will consider EoS's from other sources, in particular ref. [8]. We test 1000 of these models and compare results to the observation upper limit on a neutron star mass. We then consider possible ways this observational data constrains the theoretical models. The methods used for numerical solution to differential equations are presented in the appendix.

2 White Dwarves

2.1 Equilibrium States of a Star

First, we need to develop the equations that describe the equilibrium state of the star. To determine the critical mass, we have to consider the possible states of the star that are at an equilibrium between the degeneracy pressure and the gravitational potential inward. First, we assume the star is spherically symmetric, so we can consider the mass and density of the star depend only on the radial coordinate r. We will consider a small volume $dV = Adr$ at radius r, with mass $dm = \rho(r)dV$ where ρ is the density. Because of the spherical symmetry, the gravitational force on dm arises only from the mass that is interior to it,

$$dF_g = -\frac{GM(r)\rho(r)dV}{r^2} \tag{2.1}$$

where $M(r)$ is total mass of the star up to the radius r, and G is Newton's constant. The mass $M(r)$ can represented as

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \tag{2.2}$$

or, in differential form as

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{2.3}$$

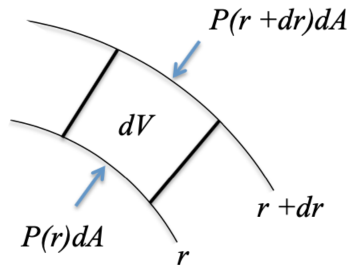

Figure 2.1: Forces on a mass element of the star.

where $\rho(r)$ is the density at a particular radius.

Next, we consider that F_p, the outward pressure is due to the pressure difference from the inside versus the outside.

$$dF_p = P(r)dA - P(r + dr)dA, \tag{2.4}$$

where dA is the area of where the pressure acts upon. Since

$$P(r + dr) = P(r) + \frac{dP}{dr}dr + \cdots \tag{2.5}$$

we find, to first order,

$$dF_P = -\frac{dP}{dr}drdA = -\frac{dP}{dr}dV \tag{2.6}$$

For the system to be in equilibrium, the inward gravitational force and the outward force due to the pressure must be equal and opposite. This produces

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}. \tag{2.7}$$

This equation along with the mass differential equation represent the equilibrium of the star. To complete the description, we must also find the relation between the pressure and the density. This is the so called equation of state. This represents the matter in the star and how it interacts within the star. Given $P(\rho)$, we can transform eq. (2.7) into an equation for $\rho(r)$ (and $M(r)$) using the Chain Rule:

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr}. \tag{2.8}$$

Then

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1} \frac{GM(r)\rho(r)}{r^2} \tag{2.9}$$

Along with eq. (2.3), we now have a coupled set of differential equations that give the density profile $\rho(r)$ and the total mass within a given radius, $M(r)$.

Before discussing the equation of state for a white dwarf, let's consider how we can solve these equations. We specify initial conditions at $r = 0$, where $M(0) = 0$ and the density has some value, $\rho(0) = \rho_c$, which we can vary.[1] We numerically integrate the equations from $r = 0$ outward, stopping when ρ reaches 0 and we reach the edge of the star. The value of r and $M(r)$ at this point give the equilibrium radius and total mass of the star, respectively. As we increase the initial value ρ_c, the overall equilibrium mass will increase, but at some point a maximum will be reached. This maximum value is the maximum mass that can be supported in equilibrium by the degeneracy pressure.

[1]It is clear that we will need to avoid the singularity in eq. (2.9) at $r = 0$. This is a technical issue we will discuss in the Appendix

2.2 Equation of State for a White Dwarf

To consider an equation of state for a white dwarf, we assume that the star is collection of protons, neutrons and electrons. Since electrons have a mass much smaller than of neutrons and protons ($\approx$ 1800 times smaller), the mass of the star is primarily due to nucleons. In a white dwarf, the environment can be described as a degenerate Fermi Gas. Degenerate Fermi gases can be defined as a gas of non-interacting fermions at a very low temperature. In this system, nearly all of the lowest possible states for the particles are filled. We can consider a white dwarf and a neutron star as a low temperature because the degenerate Fermi gas badly violates the condition for an ideal gas, that [10]

$$\frac{V}{N} >> v_Q \tag{2.10}$$

where V is the volume of the cube, N is the number of particles and v_Q is the quantum volume as defined by the de Broglie wavelength,

$$p = \frac{h}{\lambda} \tag{2.11}$$

In other words, the distance between the particles is considerably less than their de Broglie wavelengths. This means we cannot consider it an ideal gas.

In a white dwarf, the system is primarily composed approximately of the same amount of protons and electrons, so we can assume the electrons can move freely because the system is electrically neutral. Even though the temperatures of a white dwarf do not seem low (surface temperatures in a white dwarf are on the order of 10^5 K) it is reasonable to approximate $T = 0$ because the Fermi momentum (the magnitude of the largest momentum in the system) is much larger than $k_B T$. So thermal fluctuations will only rearrange electrons near the Fermi surface and overall the state of star remains essentially in the ground state all the time. To determine the pressure, we have to look at our total energy equation that we derived and see that it only depends on the volume because the electrons will only be filling the lowest possible states. So to begin ,we use a model where the white dwarf is made of N protons and N electrons.The mass is virtually only the protons and neutrons, and the electrons are treated as quasi-free as described earlier. First, we assume the mass density to be

$$\rho = n_N m_N, \tag{2.12}$$

and the electron density

$$\begin{aligned} n &= \alpha n_N \\ &= \frac{\alpha \rho}{m_N}. \end{aligned} \tag{2.13}$$

where $n_N$is the number density of nucleons, m_N is the mass of a nucleon α is the ratio of electrons to nucleons in the star. For our purposes, we will use $\alpha = 0.5$.

Now we know the degeneracy pressure in the white dwarf will be from the electrons initially so we can treat our pressure equation as a collection of N electrons in volume V at zero temperature (since it is a degenerate Fermi gas).

We now will revisit our thermodynamic identity

$$P = -\frac{\partial E}{\partial V}\Big|_N \tag{2.14}$$

which tells us we need to determine the total energy of all the electrons to determine the pressure. We know each energy level is filled by electrons, one for each spin projection, so our total energy (in the one dimensional case) is

$$E = 2\sum_{n=1}^{N/2} E_n. \tag{2.15}$$

Next, we will move to three dimensions. In cases, where N is large, summations can be approximated by integrals. A standard derivation [2] gives us the number of states with momentum components from (p_x, p_y, p_z) to $(p_x+dp_x, p_y+dp_y, p_z+dp_z)$ as

$$2V\frac{d^3p}{(2\pi\hbar)^3}. \tag{2.16}$$

By integrating this from 0 to p_f, we can produce equations for the total number of electrons

$$N = 2V\int_0^{p_f}\frac{d^3p}{(2\pi\hbar)^3}, \tag{2.17}$$

and the total energy

$$E = 2V\int_0^{p_f}\frac{d^3p}{(2\pi\hbar)^3}E(p). \tag{2.18}$$

Evaluating for the total number of electrons by switching the spherical coordinates to evaluate the momentum over all angles in momentum space,

$$N = (2V)\frac{4\pi p_f^3}{3(2\pi\hbar)^3}. \tag{2.19}$$

With this, after rearrangement

$$p_f = \hbar(3\pi^2 n)^{1/3}, \tag{2.20}$$

where $n = N/V$ is the number density of electrons.

At this point, we can check whether the condition for a degenerate Fermi gas is satisfied. If we evaluate $p_f c$ assuming a sphere with radius 10^6 m and $N = (1/2)(M_\odot/m_N)$ electrons, we find $p_f c \approx 3$ MeV, which can be approximated as the total energy because this is somewhat larger than the rest mass of an electron (0.511 MeV). Comparing this to $k_b T$ which is on the order of 10 eV for $T = 10^5$ K, we confirm that the condition for a degenerate Fermi gas is well satisfied.

Then to calculate the total energy, we treat particles to be relativistic so

$$E(p) = \sqrt{p^2c^2 + m^2c^4}. \tag{2.21}$$

This means the total energy is

$$E = 2V\int_0^{p_f} \frac{d^3p}{(2\pi\hbar)^3}\sqrt{p^2c^2 + m_e^2c^4}. \tag{2.22}$$

After evaluating this with a system that is primarily relativistic ($E \approx pc$),

$$E = 2V \int_0^{p_f} \frac{4\pi p^2 dp}{(2\pi\hbar)^3} pc = \frac{Vcp_f^4}{4\pi^2\hbar^3}. \tag{2.23}$$

and plugging that energy into eq. (2.14)

$$P = \frac{1}{4}(3\pi^2)^{1/3}\hbar c \left(\frac{\alpha}{m_N}\right)^{4/3} \rho^{4/3}, \tag{2.24}$$

This pressure is in the form of a polytrope. With this, we can determine $dP/d\rho$:

$$\frac{dP}{d\rho} = \frac{1}{3}(3\pi^2)^{1/3}\hbar c \left(\frac{\alpha}{m_N}\right)^{4/3} \rho^{1/3}. \tag{2.25}$$

Then after plugging this into eq. (2.7) we obtain

$$\frac{d\rho}{dr} = -\frac{3G}{(3\pi^2)^{1/3}\hbar c} \left(\frac{m_N}{\alpha}\right)^{4/3} \frac{M(r)\rho^{2/3}(r)}{r^2}. \tag{2.26}$$

Now we have our coupled differential equations relating $\rho(r)$ and $M(r)$ (see eq. (2.3)). To solve these equations, we will use numerical integration to graph a set of stable combinations of masses and radii of white dwarves. This will show the maximum possible mass of a white dwarf.

2.3 White Dwarf Solution

Fig. 2 shows the equilibrium masses and radii of the white dwarf star after solving the coupled differential equations for a white dwarf. For this, we used a range of values spaced logarithmically between 0.1 and 105 for the scaled central density and a starting scaled radius of 0.0001 (further details of numerical integration techniques and scaling of the differential equations can be found in the Appendix). We started just slightly off of $r = 0$ because r is the denominator. So we started the integration at 0.0001 and the initial mass being the total mass of a sphere of that radius. This is consistent with

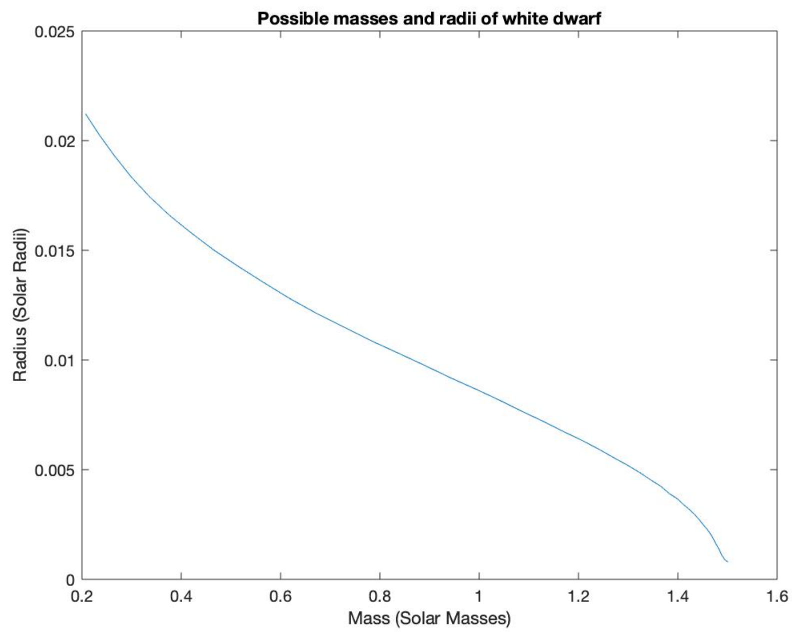

Figure 2.2: This graph shows the range of values of different mass and radii of white dwarves. This shows the Chandrasekhar limit.

the well-known Chandrasekhar mass, which is about $1.4M_{\odot}$ [4]. My solution appears higher because we used an ultra-relativistic approximation as opposed to the full energy expression.

Next, we will consider corrections to the white dwarf solution that can be used to model a neutron star and obtain a similar curve that shows the critical mass of a neutron star.

3 Neutron Stars

In applying a similar analysis to neutron stars, several important complications arise. As discussed earlier, due the strong interaction the equation of state for a dense ball of neutrons is not well known, and we must appeal to various models. In addition, a neutron star is so compact and massive that the spacetime curvature is significant, and corrections to the structure equations from Special and General Relativity are necessary. The first order corrections to the equilibrium equation were first worked out by Tolman [11] and by Oppenheimer and Volkoff [9]; these take the form

$$\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2r^2}\left[1+\frac{P(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{M(r)c^2}\right]\left[1-\frac{2GM(r)}{c^2r}\right]^{-1} \quad (3.1)$$

The first two terms in square brackets represent special relativistic corrections; the third arises from general relativity.

Note that all the correction terms are positive, so they have the effect of increasing the effect of the gravitational interactions in the star. This means they should *reduce* the maximum mass that can be supported for a given central pressure. We will investigate the numerical effects of these corrections on the solutions below.

Due to the large kinetic and potential energies of the particles in a neutron star, these energies will contribute to the total mass of the star, so the density is now represented by

$$\rho = \frac{\epsilon}{c^2} \tag{3.2}$$

As a first attempt, we can try the equation of state for a degenerate gas of neutrons rather than electrons. Of course, this neglects the strong interactions between neutrons and will not provide a realistic estimate of the maximum mass, but we should reproduce the mass limit obtained by Oppenheimer and Volkoff [9].

Looking ahead, many of the nuclear matter equations of state in the literature are given in such a way that the relation $\epsilon(P)$ can be easily inferred. In this case, there is no reason to switch from dP/dr to $d\epsilon/dr$ in eq. (3.1) using the earlier Chain Rule trick. We can just integrate directly in terms of P, since $P = 0$ also defines the edge of the star. The other equation in the set becomes, writing $\rho = \epsilon/c^2$,

$$\frac{dM}{dr} = \frac{4\pi}{c^2} r^2 \epsilon \tag{3.3}$$

Here we use the equation of state to write ϵ in terms of P at each step of the integration. The rescaling of these equations for numerical stability is much the same as before. ϵ and P have the same units, which we take to be GeV/fm^3 (see section 6.2).

To check the Oppenheimer-Volkoff calculation, it is convenient to just use a function that fits the degenerate neutron gas equation of state. This is described in section 6.3. The results for equilibrium masses and radii using this EoS are shown in figure 3.1. The maximum mass obtained is seen to be $M_{max} \approx 0.78 M_\odot$ with a corresponding radius of $R \approx 10.5$ km, in good

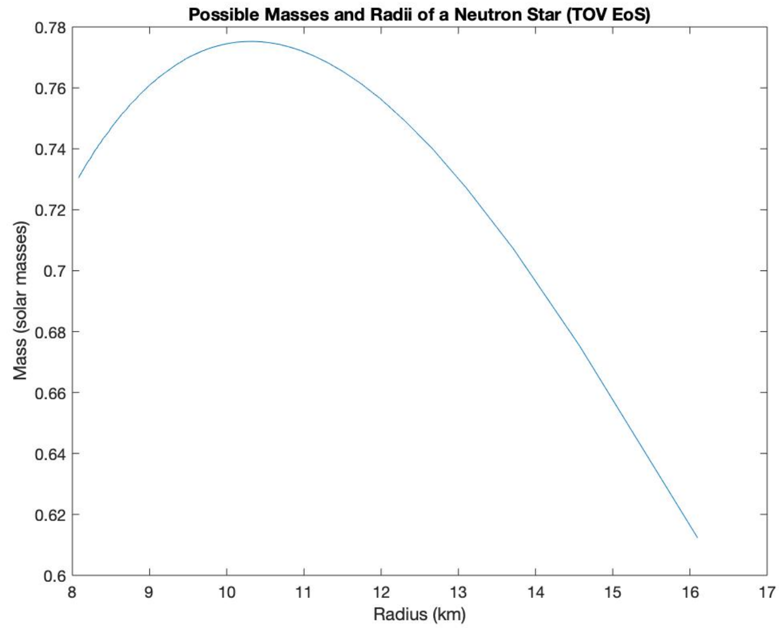

Figure 3.1: TOV solution for neutron star with calculated EoS.

agreement with ref. [9].

4 The nEoS Equations of State

Next, we study the family of nuclear matter equations of state given recently in ref. [8]. This is a new proposal for "minimally constrained" EoS's, which are intended to be as unbiased as possible.[1] They have been constructed by requiring that they agree with strong interaction theory in the regimes where reliable calculations are possible, and are otherwise essentially random, subject only to the further general constraints of causality and monotonicity. Thus these functions represent a sampling of *all possible* equations of state that are consistent with known principles. Several thousands of possible EoS's have been produced by the nEoS collaboration, and any number of others can be produced using their computational tools.

The constraints from strong interaction physics arise at very high energies and densities, where QCD perturbation theory is well defined, and at low energies and densities, where the symmetries of the strong interaction allow the construction of predictive effective field theories, an approach known as Chiral Perturbation Theory. The intermediate region is unconstrained except for the requirement of causality. The region in (ϵ, P) space that is consistent with these requirements is show in Fig. 4.1. The shaded region is then populated with points randomly to generate the equations of state.

Note that the energy/density region most relevant for neutron start struc-

[1] For example, EoS's that have been constrained by application to neutron stars using the TOV equation cannot be reliably used to test modified theories of gravity, since their development was based on general relativity.

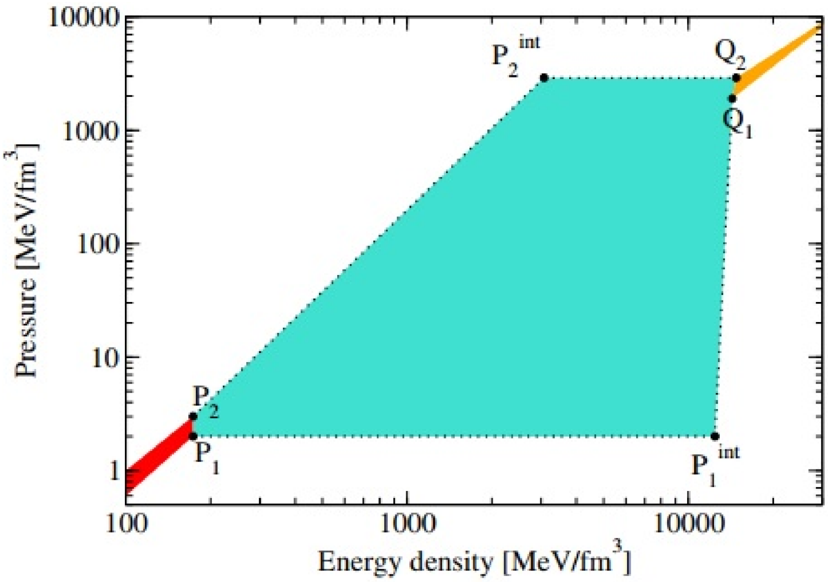

Figure 4.1: Region consistent with causality, as well as perturbative QCD (upper right) and Chiral Perturbation Theory (lower left). Figure taken from ref. [8].

ture actually lies mostly in the shaded region between the two anchor points. Our goal here is to show how much of this "equation space" is consistent with current mass limits on neutron stars.

4.1 Calculations

We chose a sampling of 1000 of these equations, a few of which are shown in Fig. 4.2. The convergence in the upper right corner shows the matching to perturbative QCD, and the lower left to Chiral Perturbation Theory.

These equations of state are given in the form of table (ϵ, P) values. To extend the relations to other values we will interpolate linearly between the given values. Given a value of P for which we want $\epsilon(P)$, we first find the

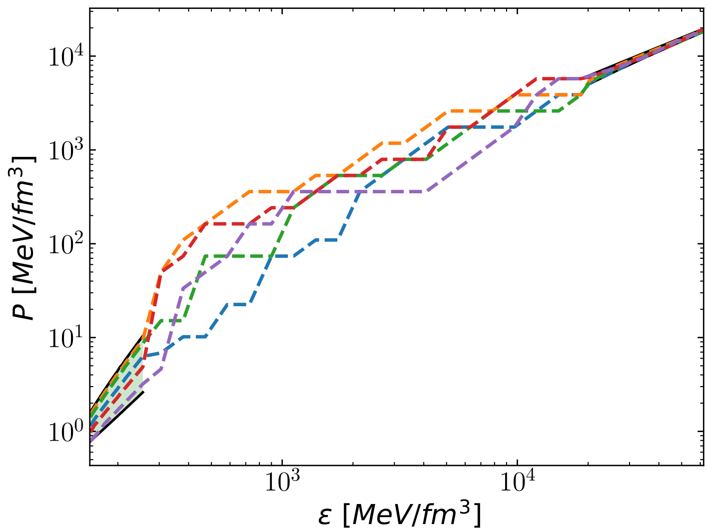

Figure 4.2: Sample EoS satisfying all constraints. The low-density regime is constrained by next-to-leading-order chiral potentials, with momentum cutoff $\Lambda = 450$ and 600 MeV. Figure taken from ref. [8].

bracketing values of P given in the file, that is P_1 and P_2 such that

$$P_1 < P < P_2 \tag{4.1}$$

The values of ϵ corresponding to P_1 and P_2 are ϵ_1 and ϵ_2, respectively. Then the interpolated value of ϵ is (see Fig. 4.3)

$$\epsilon(P) = \epsilon_1 + mP \tag{4.2}$$

where m is the slope of the interpolating line,

$$m = \frac{\epsilon_2 - \epsilon_1}{P_2 - P_1} \tag{4.3}$$

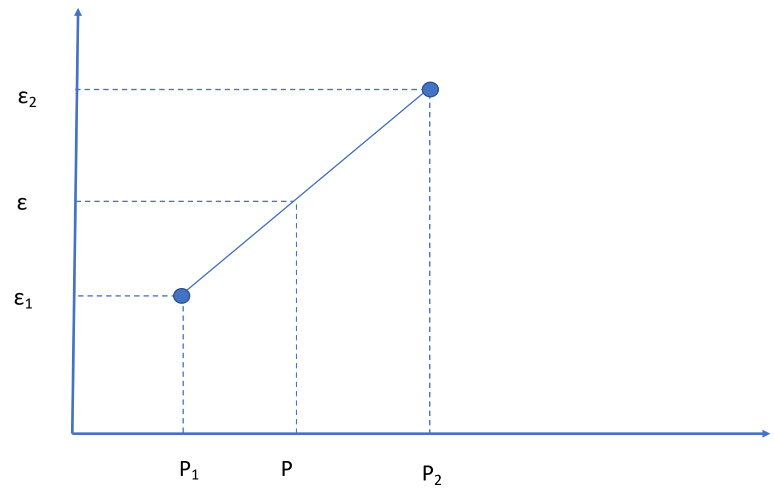

Figure 4.3: Interpolation of nEoS data.

We can now calculate the mass limits and corresponding radii just as before. A typical example is given in

Fig. 4.4. We see that the mass limit is significantly larger than the TOV limit based on a degenerate neutron gas, and indeed is in the neighborhood of the largest neutron star masses observed. We will discuss the observational data further below.

We then calculated the limiting mass for all 1000 of the nEoS models, to get a sense of the range of possible results. This was automated by increasing the starting with a low central pressure (recall that we are integrating in terms of P now) and increasing it until the equilibrium mass obtained starts

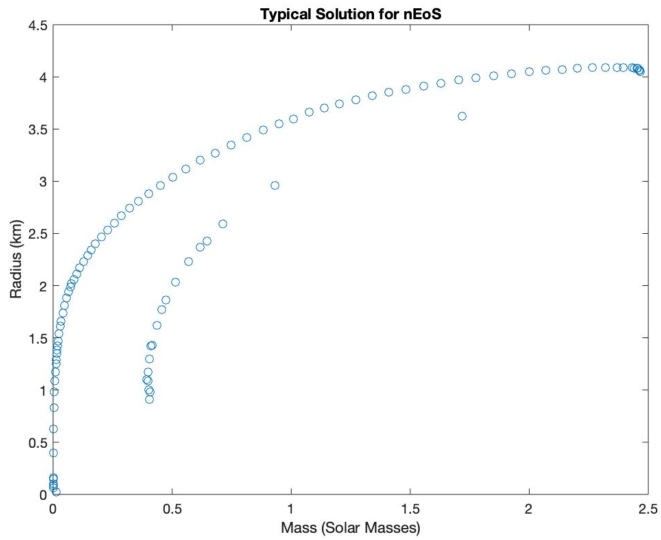

Figure 4.4: Typical solution for a nEoS equation of state.

to decrease; the maximum value is then saved, along with the corresponding radius. Results for these models are shown in Fig. 4.5, from which we see that the limiting masses run from about $(1 - 2.8) \times M_{\odot}$ for these models.

4.2 The Observational Data

Precision data on neutron stars has been accumulating at an accelerating rate, with many discoveries coming in the past 15 years. The current situation is summarized in a recent review article [12]. We now have precise[2] masses for more than 35 neutron stars spanning a range from $(1.17 - 2.27)M_{\odot}$. The current record holder is a millisecond pulsar in PSR J2215+5135 with mass $2.27^{+0.17}_{-0.15} M_{\odot}$ [13]. There are many additional measurements with larger error bars, but which may suggest even larger maximum mass values. A fairly

[2]Defined as: 1σ uncertainty is less than 15% of the measured value.

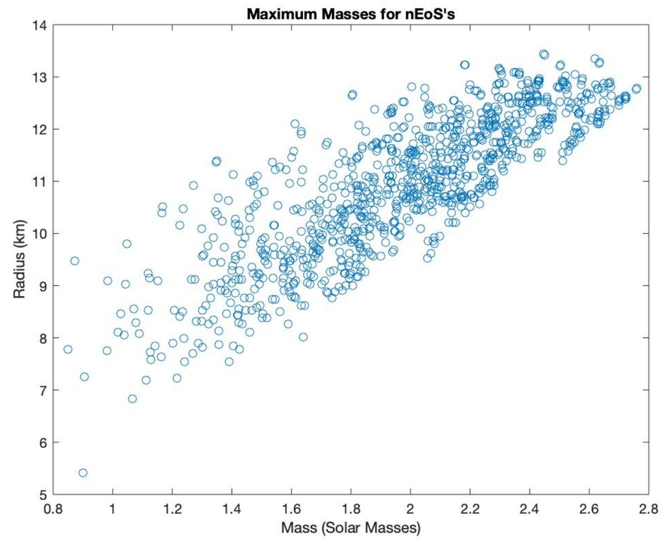

Figure 4.5: Maximum masses and corresponding radii for 1000 equations of state given in ref. [8].

complete summary of the mass observations is shown in Fig. 4.6.

In addition, accurate radii are known for more than a dozen neutron stars, with values in the range $9.9 - 11.2$ km.

With this data, we can eliminate solutions that have critical masses that are below $2.10 M_\odot$. After removing these solutions, we find that roughly 43.8% percent of the EoS's remain consistent with observational data. If the recent candidate with mass $2.27 M_\odot$ is confirmed, the fraction of consistent models drops to about 26.8%.

In general, the models the produce larger mass limits tend to have higher pressures for a given energy density, that is, run through the upper part of the region allowed by causality in Fig. 4.1. However, the equations of state are not always monotonic, and can have rapid changes in slope, it is difficult to make clear generalizations about what features of the equations tend to produce large (or small) limiting masses.

As more data on neutron stars becomes available, we can look into constraining the possible EoS's further. Especially useful would be examples where the mass and radius can *both* be determined to good accuracy, since the models predict the relation between them. The current situation in this regard is shown in Fig. 4.7 – in cases where both values are measured, the accuracy is not yet sufficient to provide strong constraints on the nuclear physics.

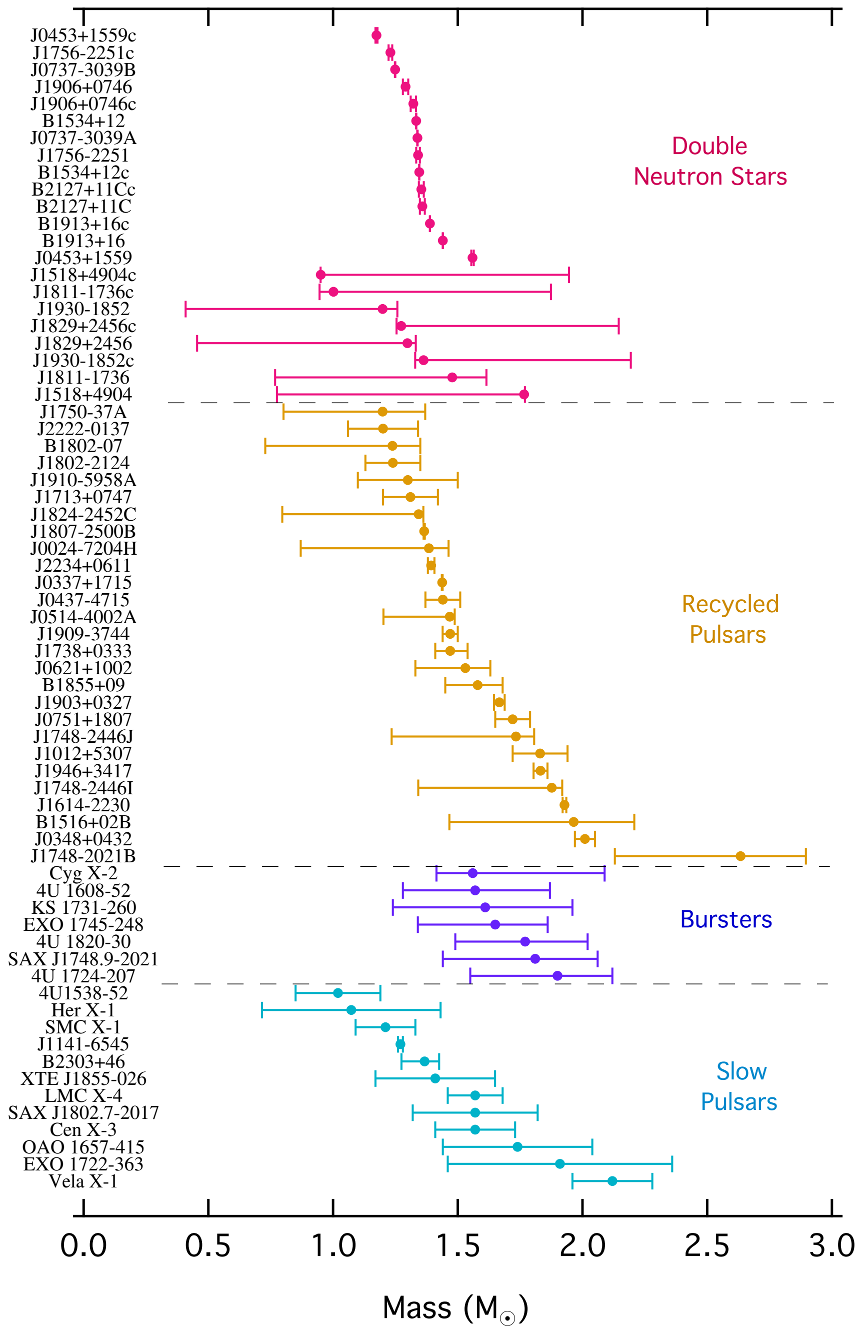

Figure 4.6: Summary of recent measurement of neutron star masses. Double neutron stars (magenta), recycled pulsars (gold), bursters (purple), and slow pulsars (cyan) are included. Figure taken from ref. [12].

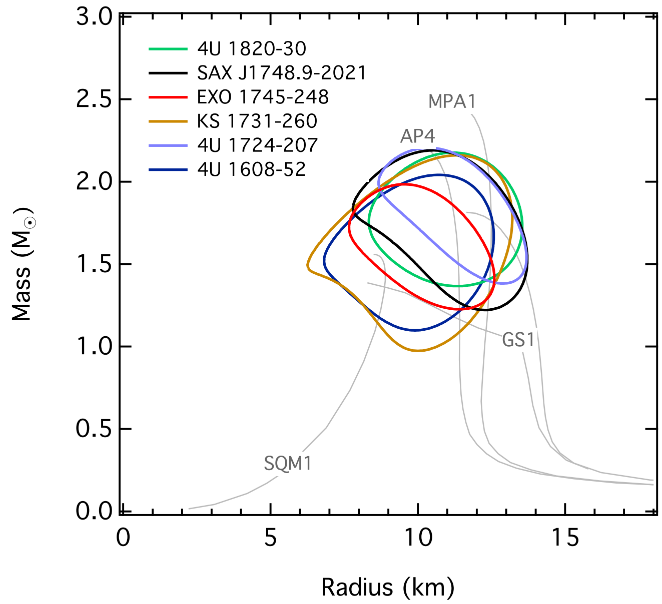

Figure 4.7: Representative neutron star mass-radius relations at the 68% confidence level. The light grey lines show mass-relations corresponding to a few representative equations of state. Figure taken from ref. [12].

5 Conclusions

In this thesis I have investigated how calculations of neutron star equilibrium states can constrain theoretical proposals for the neutron matter equation of state. We investigated a set of EoS's proposed by [8], and further constrained them by solving for the maximum mass that can be supported and comparing to the observed properties of neutron stars. Improvements in the data, in particular the detection of neutron stars whose masses and radii can be determined accurately, will further constrain these relations.

6 Appendix

6.1 Numerical Methods for Integration

The Runge-Kutta method is a technique used to numerically integrate differential equations [14]. It is based on Euler's method that can also be used to numerically integrate differential equations. Euler's method can be derived from the equation for the slope of a line. It is as follows:

$$y(x + \epsilon) = y(x) + \epsilon f(x, y) \tag{6.1}$$

where ϵ is the distance between two points and $f(x, y)$ is the derivative of the interpolated line between x and $x + \epsilon$. By using small step sizes, we can iteratively create an approximation of a given function. This method still proves to converge slowly to the actual solution so the higher order Runge-Kutta methods can be used. In Euler's method, it requires very small steps to remain accurate which increases the computational cost. So the Runge-Kutta method provides a more accurate slope with a larger step size. This method takes intermediate points in between each step to create a weighted average to determine the next point in a function. In my solutions, I used the 4th order Runge-Kutta method(sometimes referred as RK4) which is as follows.

$$k_1 = \epsilon f(x, y) \tag{6.2}$$

$$k_2 = \epsilon f(x + \epsilon/2, y(x) + k_1/2) \tag{6.3}$$

$$k_3 = \epsilon f(x + \epsilon/2, y(x) + k_2/2) \tag{6.4}$$

$$k_4 = \epsilon f(x + \epsilon, y(x) + k_3) \tag{6.5}$$

$$y(x + \epsilon) = y(x) + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + \mathcal{O}(\epsilon^5) \tag{6.6}$$

k_1 is just the slope at the beginning of the interval. k_2 is the slope found at the midpoint of the step using y and k_1. k_3 is the slope based on the midpoint of the interval using y and k_2. k_4 is the slope at the end of the interval using y and k_3. In the case of our calculations, we ignore the 5th and higher order corrections. In the case of our equations of state for the white dwarf, there are coupled differential equations. With this, we had to use the Runge-Kutta equation twice because both the mass density and the mass where changing with each step in the radius. Furthermore, the mass density and mass were both dependent on each other, so they must be determined after each step. In the end, we will be using a large range of initial central densities to determine many possible combinations of mass and radii of white dwarfs. One issue that arises with this is that there is a singularity at r = 0, so we start the integration at $r = \epsilon$. This is also means that the starting mass will also not be 0 but the mass of a sphere with radius ϵ.

6.2 Scaling the Equations

One issue that may arise is that the numbers being using to integrate our differential equations are very large or very small. This can cause rounding errors as the equations are being integrated. So scaling our equations to carry out numerical integrations in terms of variables that aren't too large or small is necessary. To do this, we introduce dimensionless variables

$$r = R_0\bar{r}, \quad \rho = \rho_0\bar{\rho}, \quad M = M_0\bar{M} \tag{6.7}$$

which will replace the corresponding variables. Then we will determine the constants in front of them but first making an arbitrary distinction for one of the scaling factors and then setting the other two values to cancel out many of the remaining constants.

ρ_o will be set arbitrarily to

$$\rho_0 = \frac{n_0 m_N}{\alpha}, \tag{6.8}$$

where n_0 is defined as

$$n_0 = \frac{m_e^3 c^3}{3\pi^2 \hbar^3}, \tag{6.9}$$

$\alpha = 0.5$ (the fraction of protons to nucleons) and m_N is the mass of a nucleon. Then, we choose the other two constants to make the factors in parentheses equal to 1

$$R_0 = \left[\frac{(3\pi^2)^{1/3}\hbar c}{12\pi G}\left(\frac{\alpha}{m_N}\right)^{4/3}\frac{1}{\rho_0^{2/3}}\right]^{1/2}. \tag{6.10}$$

$$\frac{d\bar{M}}{d\bar{r}} = \left[\frac{4\pi R_0^3 \rho_0}{M_0}\right]\bar{r}^2\bar{\rho} \tag{6.11}$$

$$\frac{d\bar{\rho}}{d\bar{r}} = -\left[\frac{3GM_0}{R_0\rho_0^{1/3}(3\pi^2)^{1/3}\hbar c}\left(\frac{m_N}{\alpha}\right)^{4/3}\right]\frac{\bar{M}\ \bar{\rho}^{2/3}}{\bar{r}^2} \tag{6.12}$$

Then after numerically integrating these equations, they can be restored back to their physical units by simply multiplying the solutions by the constants. Similar methods will be used for the neutron star models.

6.3 Neutron Degenerate Gas EoS

For the TOV equation, we need to derive the energy density as a function of pressure. To start, the energy density obtained by evaluating the integral in eq. (2.22), is [15]

$$E = Vn_0 m x^3 f(x), \tag{6.13}$$

where

$$f(x) = \frac{3}{8x^3}\left\{x(1+2x^2)(1+x^2)^{1/2} - \ln[x + (1+x^2)^{1/2}]\right\} \tag{6.14}$$

and

$$x \equiv \frac{p_f}{m} = \left(\frac{n}{n_0}\right)^{1/3} \tag{6.15}$$

where

$$n_0 \equiv \frac{m^3}{3\pi^2} \tag{6.16}$$

In other words,

$$E = \epsilon_0 V \tilde{f}(x), \tag{6.17}$$

where

$$\tilde{f}(x) = \frac{1}{8}\left\{x(1+2x^2)(1+x^2)^{1/2} - \ln[x + (1+x^2)^{1/2}]\right\} \tag{6.18}$$

and

$$\epsilon_0 = \frac{m^4}{\pi^2} \tag{6.19}$$

These formulae are given in "natural" units where $\hbar = c = 1$. To restore the physical units, note that ϵ_0 should have units of $energy/length^3$. To account for this, we restore the $\hbar$s and cs to obtain

$$\epsilon_0 = \frac{m^4 c^8}{\pi^2 (\hbar c)^3} \tag{6.20}$$

Numerically, for neutrons,

$$\epsilon_0 = \frac{(0.939 \text{ GeV})^4}{\pi^2 (0.1973 \text{ GeV fm})^3} = 10.256 \text{ GeV/fm}^3 \tag{6.21}$$

The energy density then is

$$\epsilon = \frac{\epsilon_0}{8} \left\{ x(1+2x^2)(1+x^2)^{1/2} - \ln[x + (1+x^2)^{1/2}] \right\} \tag{6.22}$$

Now, we calculate the pressure with the adiabatic thermodynamic identity,

$$P = -\frac{\partial E}{\partial V}\bigg|_N \tag{6.23}$$

Notice that

$$x = CV^{-1/3} \tag{6.24}$$

so

$$\frac{dx}{dV} = -\frac{1}{3} C V^{-4/3} = -\frac{x}{3V} \tag{6.25}$$

The result turns out to be

$$P = \frac{\epsilon_0}{24}\left\{(2x^3 - 3x)(1 + x^2)^{1/2} + 3\ln[x + (1 + x^2)^{1/2}]\right\}. \qquad (6.26)$$

Now, ideally , we would solve this equation for $x(P)$ and plug it into the energy to give the energy as a function of P. This task proves difficult so instead we generate a table of (ϵ, P) values, for a reasonable range of x values, and fit this curve. Taking $0.05 \leq x < 10$, we find

$$\epsilon/\epsilon_0 = -0.336936x^{4/5} + 1.69299x^{3/5} + 3.03176x \qquad (6.27)$$

where $x = P/\epsilon_0$. Note that we include the both polytropes relating to the non-relativistic and ultra-relativistic limits, as well as a linear term for a better fit. After plugging in $x = P/\epsilon_0$ and combining all the constants gives:

$$\epsilon(P) = -0.536714P^{4/5} + 4.29582P^{3/5} + 3.03176P \qquad (6.28)$$

In the TOV equation, this gives a maximum mass of about $0.78M_\odot$.

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