Spring 4-11-2016

Using Precise Measurements of Muon g-2 to Constrain New Physics

Evan M. Heintz
Otterbein University, evan.heintz@otterbein.edu

USING PRECISE MEASUREMENTS OF MUON G-2
TO CONSTRAIN NEW PHYSICS

Otterbein University
Department of Physics
Westerville, Ohio 43081
Evan M. Heintz

11 April 2016

Submitted in partial fulfillment of the requirements for
graduation with Honors

David G. Robertson, Ph.D. Project Advisor	__________________________ Advisor’s Signature
Uwe Trittmann, Ph.D. Second Reader	__________________________ Second Reader’s Signature
Wendy Sherman Heckler, Ph.D. Honors Representative	__________________________ Honors Rep’s Signature

Acknowledgements
I would like to thank my advisor, Dr. David Robertson, for his encouragement, never-ending enthusiasm, and countless hours of teaching and working to ensure my success on this project. I would further like to thank Jeffrey Heintz and Jill Jones-Heintz for their never-ending belief in me and for encouraging my love of math and physics from an exceptionally young age. This project would not have been possible without their support.

Abstract

We study the contributions to the anomalous magnetic moment of the muon from theories beyond the Standard Model, specifically supersymmetry. Results found at Brookhaven National Laboratory in 2001 during the E821 experiment indicate that current theories may not fully account for all of the interactions between the muon and fundamental gauge bosons. We re-derive Dirac's famous result of $g = 2$. We then reproduce the one loop contributions from electroweak theory to the anomalous magnetic moment, $a_\mu = (g-2)/2$, of the muon. We then use these calculations as a template for theories that go beyond the Standard Model. We show illustrative results for minimal supersymmetry in scenarios with Planck-scale supersymmetry breaking (MSUGRA), and determine parameter ranges that would be consistent with the a_μ discrepancy. In this way a sensitive probe like a_μ may be used to constrain new physics.

Contents

1 Introduction

The first half of the 20th century was a period of perhaps the greatest revolution in any scientific field since Isaac Newton first published his *Principia* in 1687. In 1905, Albert Einstein published his papers on special relativity, which connected the ideas of space and time. In addition, in the first three decades of the 20th century, great physicists were beginning to work together to form the field of non-relativistic quantum mechanics, which was essentially completed by 1927.

Much of physics, however, is dedicated to uniting all of the subfields of physics into one great unified theory. Therefore, the next step for physicists was to make quantum mechanics consistent with special relativity. This theory combining the two fields, known as relativistic quantum field theory (RQFT), was formulated in the 1940s with the help of great names in physics, like Born, Dirac, Jordan, Heisenberg, and Pauli. Since then, RQFT has provided a spectacularly successful framework for describing elementary particles and their interactions at accessible energies.

One of RQFT's earliest triumphs was when Dirac proved that the g-factor equaled two for the spin angular momentum contribution to the magnetic moment of a fermion. The g-factor is a proportionality constant found in the equation for the magnetic moment, μ, of a particle with spin $\vec{S}$:

$$\vec{\mu} = g\frac{e}{2m}\vec{S}$$

where e is the charge of an electron, m is the mass of the particle, and $\vec{S}$ is the spin vector for the particle. This result had been inferred from atomic spectra which showed the small shifting of energy levels due to the interaction of the magnetic moment of an electron with its environment. It had yet to be explained, though, by a calculation until Dirac's success. However, experiments later showed that the g-factor did not, in fact, equal two, but instead, equaled two plus a small correction. Then, in 1948, Julian Schwinger produced another major triumph for RQFT in his calculation of correction, known as the the "anomalous" magnetic moment $(g-2)$ of the electron and muon. Since then, precise measurements of $a_\mu = (g-2)/2$ of fermions, specifically muons, have been compared to precise calculations done by theorists for any discrepancies. Since a_μ gets contributions from all particles that couple directly or indirectly to the fermion, a difference between the two could be an important clue to new physics, i.e., particles and interactions not yet incorporated into the Standard Model.

In 2001, the E821 experiment at Brookhaven published the most precise measurement ever done of a_μ of the muon. The experiment found a 2.7σ discrepancy between the theoretical and experiment value, and since then, improved calculations have raised that to a $3.3-3.6\sigma$ difference. This discrepancy is one of the strongest indications so far that there is physics beyond the Standard Model.

Our aim in this thesis is to take the E821 result and the improved theoretical values and constrain the new particle or particles that may be responsible for this discrepancy. We will focus on supersymmetry as an example of possible new physics, and show how the experimental result can be used to constrain its structure. However, to set the stage and show how these calculations are done, we will first reproduce the original one loop corrections made to a_μ from the electroweak theory.

2 Background

2.1 Quantum Field Theory's Early Success

In 1928, Paul Dirac developed a relativistic wave equation, the Dirac equation, that naturally incorporated spin and correctly predicted antiparticles. One of its main features was that it gave the correct ratio between the spin and magnetic moment which had been inferred from atomic spectra. Physicists had long been confused by the g-factor for the spin magnetic dipole moment of the electron which was twice as large as the g-factor for the orbital magnetic moment. This was difficult to understand because it meant that the electron could not be considered a rotating ball of charge. Therefore, its spin had to be considered more as an intrinsic quality rather than something we could observe, like the rotation of the Earth. However, Dirac's relativistic wave equation was able to naturally produce this result with some slight manipulations of the equation[1]. I will now show a derivation of this proof, in order to show one of the early successes of RQFT.

The Dirac equation for a free particle reads:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad , \quad \mu = 0,1,2,3 \tag{1}$$

where the derivative four-vector is defined as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

To obtain this equation, Dirac started from:

$$p^\mu p_\mu - m^2 = 0$$

which is the energy-momentum relation:

$$E^2 = p^2 + m^2$$

in four-vector form. Here, E is the energy, p is the momentum, and m is the mass of the particle.

(We use units in which $c = \hbar = 1$ throughout.)

His goal was to try and make quantum mechanics consistent with special relativity. Therefore, since in relativity space and time are equivalent, Dirac decided to treat them on the same footing in quantum mechanics as well. The nonrelativistic Schrodinger equation:

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{\nabla^2}{2m} + V(x)\right]\Psi$$

does not work in this scenario since space and time are not equivalent in this equation (first derivative for time is proportional to the second derivative for space). Therefore, he tried factoring this relationship into:

$$p^\mu p_\mu - m^2 = (\beta^k p_k + m)(\gamma^\lambda p_\lambda - m) = \beta^k \gamma^\lambda p_k p_\lambda - m(\beta^k - \gamma^k)p_k - m^2$$

where β and γ are eight constants we need to determine. In the second term we let $\lambda = k$ since it is a dummy variable. Since no terms can be linear in p, we must conclude that $\gamma^k = \beta^k$. This means all that is left to determine is what γ is such that:

$$p^\mu p_\mu = \gamma^k \gamma^\lambda p_k p_\lambda$$

This relation is only true if $(\gamma^0)^2 = 1$ and $(\gamma^i) = -1$. It is here that Dirac had an inspiration that these γ constants must be matrices. Therefore, γ^μ represents the four gamma matrices. The matrices must satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

so that $\gamma^k \gamma^\lambda p_k p_\lambda = p^\mu p_\mu$. Here:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the Minkowski metric. The brackets around our gamma matrices are called the anticommutator:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu$$

We find that the smallest matrices that satisfy these are 4×4 matrices. For convenience, we will choose a particular representation of these gamma matrices, known as the Dirac representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad i = 1, 2, 3$$

where each section of the above matrices represents a separate 2×2 matrix. Here, σ^i are the Pauli matrices defined as:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Finally, since the gamma matrices are 4×4 matrices, we must then conclude that ψ, the wavefunction, is a four component column matrix, known as a bispinor. In the Dirac representation, we can split it into two components:

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

where ϕ is the top two components and χ is the bottom two components.

Now, we want to show that $g = 2$. We let the particle interact with an electromagnetic field since we know the magnetic potential energy from this interaction is $U = -\vec{\mu} \cdot \vec{B}$ and g is contained in μ. To get the correct form of the Dirac equation for an electron interacting with an electromagnetic field, we replace ∂_μ with the covariant derivative:

$$D_\mu = \partial_\mu - ieA_\mu$$

Here, A^μ is the four vector electromagnetic potential:

$$A^\mu = (\phi, \vec{A})$$

where ϕ is the scalar potential and $\vec{A}$ is the magnetic vector potential. Therefore, Dirac's equation now reads:

$$(i\gamma^\mu D_\mu - m)\psi = 0 \tag{2}$$

We can multiply this by $i\gamma^\nu D_\nu + m$, resulting in:

$$-(\gamma^\mu \gamma^\nu D_\mu D_\nu + m^2)\psi = 0$$

Our definitions of the gamma matrices earlier imply:

$$\gamma^\mu \gamma^\nu = \frac{1}{2}(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) \qquad \frac{1}{2}[\gamma^\mu, \gamma^\nu] = -i\sigma^{\mu\nu}$$

where we define:

$$\sigma^{0j} = i\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} \qquad \sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad \sigma^{\mu\nu} = -\sigma^{\nu\mu}$$

and ϵ^{ijk} is the Levi-Civita symbol:

$$\epsilon^{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2) \\ -1 & (i,j,k) = (3,2,1),\ (2,1,3),\ (1,3,2) \\ 0 & i=j,\ j=k,\ i=k \end{cases}$$

Furthermore, the notation of $[\gamma^\mu, \gamma^\nu]$ indicates the commutator:

$$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$$

With these substitutions, we arrive at:

$$-(D_\mu D^\mu - i\sigma^{\mu\nu}D_\mu D_\nu + m^2)\psi = 0$$

Since our indices are interchangeable here and $\sigma^{\mu\nu}$ is antisymmetric under exchange of its indices, we can rewrite this as:

$$-(D_\mu D^\mu - \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu] + m^2)\psi = 0$$

Finally, one can show that $i[D_\mu, D_\nu] = eF_{\mu\nu}$ (Appendix A), with the electromagnetic field strength tensor defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

However, since we know the derivatives of the potentials are the fields, the elements of $F_{\mu\nu}$ can be related to electromagnetic fields. In fact:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The resulting equation is:

$$\left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\psi = 0 \tag{3}$$

Now, consider a weak magnetic field in the 3-direction. Since it is a weak field, any terms of order A_i^2 or greater can be ignored. Therefore, we let $A_0 = 0$, $A_1 = -\frac{1}{2}Bx^2$, and $A_2 = \frac{1}{2}Bx^1$ so that

$F_{12} = \partial_1 A_2 - \partial_2 A_1 = B$. Therefore, we find that $\partial_i A_i = 0$. The $D_\mu D^\mu$ term can be rewritten as $(D_0)^2 - (D_i)^2$. The $(D_0)^2$ term will just be $(\partial_0)^2$ since $A_0 = 0$. Therefore, we will examine the $(D_i)^2$ term first since it contains the A_i terms:

$$\begin{aligned}
(D_i)^2 &= (\partial_i)^2 - ie\left(\partial_i A_i + A_i \partial_i\right) + O(A_i^2) \\
&= (\partial_i)^2 - 2ieA_i\partial_i + O(A_i^2) \\
&= (\partial_i)^2 - 2ie(A_1\partial_1 + A_2\partial_2) + O(A_i^2) \\
&= (\partial_i)^2 - 2ieB\left(-\frac{x^2}{2}\partial_1 + \frac{x^1}{2}\partial_2\right) + O(A_i^2) \\
&= \vec{\nabla}^2 + ieB(x^2\partial_1 - x^1\partial_2) + O(A_i^2)
\end{aligned}$$

In the second step, we used the product rule since our expression is still multiplied by our wavefunction. Finally, since $-i\partial_\mu = p_\mu$ we have:

$$B(xp_y - yp_x) = \vec{B}\cdot(\vec{x}\times\vec{p}) = \vec{B}\cdot\vec{L}$$

and hence:

$$(D_i)^2 = \vec{\nabla}^2 + e\vec{B}\cdot\vec{L} + O(A_i^2) \tag{4}$$

We can take the nonrelativistic limit here to try and make a connection with nonrelativistic quantum mechanics. In the Dirac basis, the top two components contained in ϕ are much larger than the two components contained in χ. This means $\chi \to 0$ so we only have to focus on ϕ here. Acting on ϕ with $(e/2)\sigma^{\mu\nu}F_{\mu\nu}$, we obtain:

$$\frac{e}{2}\left(\sigma^{12}F_{12} - \sigma^{21}F_{21}\right) = \frac{e}{2}\sigma^3\left(F_{12} - F_{21}\right)$$

since every other term of $F_{\mu\nu}$ will be zero. However, $F_{12} = -F_{21}$ and $F_{12} = B$ as we showed earlier. This leaves us with:

$$\frac{e}{2}\sigma^3(2B) = \frac{e}{2}\vec{\sigma}\cdot 2\vec{B}$$

since $\vec{B}$ points in the third direction. Finally, we recall that $\vec{S} = \vec{\sigma}/2$ so that our final result is

$$\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = 2e\vec{B}\cdot\vec{S} \tag{5}$$

Finally, the only remaining part we have left is the $\left(\partial_0^2 + m^2\right)\phi$ term. In the nonrelativistic limit, the energy will be $E = m + E_{NR}$, the rest energy plus the part we would call the energy in nonrelativistic physics. These nonrelativistic energies will be much smaller than the rest energy and so we can factor out the rapid time dependence that comes from the constant rest energy.

Therefore, we let $\phi = e^{-imt}\Psi$. By the product rule:

$$\frac{\partial}{\partial t}\left(e^{imt}\Psi\right) = e^{-imt}\left(-im + \frac{\partial}{\partial t}\right)\Psi$$

This when used twice simplifies our equation to:

$$e^{-imt}\left[\left(-im + \frac{\partial}{\partial t}\right)^2 + m^2\right]\Psi$$

In the nonrelativistic limit, any energy contained in Ψ will be small compared to the rest energy contained in the exponent. Therefore, Ψ oscillates much more slowly than the exponential term. Therefore, we can drop the squared derivative term and arrive at:

$$\left(\partial_0^2 + m^2\right)e^{-imt}\Psi \approx e^{-imt}\left[-2im\frac{\partial}{\partial t}\Psi\right] \tag{6}$$

Putting all of our results together, canceling the e^{-imt} term, and dividing by $2m$, we are left with:

$$\left[-i\frac{\partial}{\partial t} - \frac{\vec{\nabla}^2}{2m} - \frac{e\vec{B}}{2m}\cdot\left(\vec{L} + 2\vec{S}\right)\right]\Psi = 0 \tag{7}$$

Thus, we arrive at the Schrodinger equation, as we should in the nonrelativistic limit. We recall that the potential energy is $-\vec{\mu}\cdot\vec{B}$, giving:

$$\vec{\mu} = \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right)$$

Thus, the factor of two for the spin angular momentum of the electron, or any spin $1/2$ particle, interacting with the electromagnetic field appears naturally. This result was one of the earliest triumphs for RQFT and began to show that the theory was producing correct results. Specifically, this result indicates that spin is a relativistic phenomenon.

2.2 One Loop Correction from Interactions with a Photon

By 1947, it had become evident from various experiments that the magnetic moment actually differed from two by a small amount, known as the anomalous magnetic moment, or a_μ. In 1948, Julian Schwinger published a paper[2] that provided the calculation for the one loop contribution to a_μ from quantum electrodynamics (QED). At the time, the calculation was complicated. However, the calculation becomes much more tractable using the diagrammatic methods developed by Richard Feynman.

We begin by first examining the value of $\langle p', s'|J^\mu|p, s\rangle$, the matrix element between fermion states of the four-vector electromagnetic current, where:

$$J^\mu = \left(\rho, \vec{J}\right)$$

where ρ is the charge density and $\vec{J}$ is the current density. The states $|p, s\rangle$ and $|p', s'\rangle$ are momentum and spin eigenstates ($s = \pm 1/2$). Since the Lagrangian density for QED contains the interaction term:

$$\mathcal{L}_{\text{int}} = -eJ^\mu A_\mu \ ,$$

this operator controls the interaction of the fermion with the electromagnetic field, where A^μ is the electromagnetic vector potential. To lowest order in perturbation theory:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')\gamma^\mu u(p, s)$$

where $u(p, s)$ is a plane-wave solution to the free Dirac equation:

$$(\gamma^\mu p_\mu - m)u(p, s) = 0$$

and $\overline{u} \equiv u^\dagger\gamma^0$. The dagger over the u indicates that this is the Hermitian conjugate of the bispinor u. Note that $\overline{u}$ satisfies:

$$\overline{u}(p, s)(\gamma^\mu p_\mu - m) = 0$$

The normalization of the u's is such that:

$$\overline{u}(p, s')u(p, s) = 2m\delta_{s,s'}$$

Including higher order corrections to the matrix element, we can write:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')(\gamma^\mu + \Gamma^\mu)u(p, s) \tag{8}$$

where Γ^μ includes the contributions of higher order Feynman diagrams.

Now, Lorentz invariance and current conservation imply that[3]:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p, s) \tag{9}$$

where $q = p' - p$ and $F_{1,2}$ are functions of the Lorentz invariant q^2, known as "form factors." We can learn nothing further about these form factors from Lorentz invariance and current conservation. To lowest order in perturbation theory, $F_1(q^2) = 1$ and $F_2(q^2) = 0$. We want to relate this expression to the magnetic moment of a fermion. To see this, we use an useful identity relating spinors and the gamma matrices known as the Gordon Decomposition (Appendix B):

$$\overline{u}\gamma^\mu u = \overline{u}\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u$$

Using this, the expression in eq. (9) becomes:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')\left[\frac{(p'+p)^\mu}{2m}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(F_1(q^2) + F_2(q^2))\right]u(p, s)$$

As before, we can consider a slowly moving fermion interacting with an essentially static field. Hence, we are interested in the limit: $q^\mu \sim 0$, so that:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')\left[\frac{(p'+p)^\mu}{2m}F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(F_1(0) + F_2(0))\right]u(p, s) \qquad (10)$$

Now, the first term is independent of the spin and is what we would get for a spinless particle interacting with an EM field. In that case, the Lagrangian has the form:

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* + \cdots$$

where ϕ is a scalar field and $D_\mu = \partial_\mu - ieA_\mu$ as before. Thus, the Lagrangian contains the terms:

$$\left[\phi(\partial_\mu\phi^*) - (\partial_\mu\phi)\phi^*\right]A^\mu$$

which give a Feynman rule proportional to $(p+p')^\mu$. This suggests that $F_1(0)$ is just the fermion charge (in units of e), and thus, $F_1(0) = 1$ to all orders in perturbation theory. This will be shown in detail below.

The second term, involving the spin operators, contains the magnetic moment we want. This can be shown, as above, by measuring the fermion's interaction with a static magnetic field and extracting the part of the energy proportional to the field. But, since:

$$F_1(0) + F_2(0) = 1 + \mathcal{O}(\alpha)$$

we infer that:

$$g = 2[F_1(0) + F_2(0)]\ .$$

Hence $F_2(0)$ gives the correction to $g = 2$. We could now calculate $F_2(0)$ by writing down the Feynman diagrams for the current matrix element and picking out the correct terms in the limit $q \to 0$. However, we want to first prove that $F_1(0) = 1$ to all orders. We can solve for the value of the first form factor by examining charge conservation.

2.2.1 *Determining* $F_1(0)$

Consider the matrix element of the charge operator $\hat{Q}$[3]:

$$\langle p', s'|\hat{Q}|p, s\rangle = (2\pi)^3 \cdot 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}\,' - \vec{p})\delta_{ss'} \tag{11}$$

Since $|p, s\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue 1, $\hat{Q}$ is simply:

$$\hat{Q} = \int d^3\vec{x} J^0(x)$$

Translation invariance allows us to rewrite $J^0(x)$ as:

$$J^0(x) = e^{-i\hat{p}\cdot\vec{x}} J^0(0) e^{i\hat{p}\cdot\vec{x}}$$

where $\hat{p}$ is the momentum operator:

$$\hat{p}|0\rangle = 0, \qquad \hat{p}|p, s\rangle = \vec{p}\,|p, s\rangle, \qquad \text{etc.}$$

Thus:

$$\hat{Q} = \int d^3\vec{x} e^{-i\hat{p}\cdot\vec{x}} J^0(0) e^{i\hat{p}\cdot\vec{x}}$$

and hence:

$$\begin{aligned}
\langle p', s'|\hat{Q}|p, s\rangle &= \int d^3\vec{x} \langle p', s'|e^{-i\hat{p}\cdot\vec{x}} J^0(0) e^{i\hat{p}\cdot\vec{x}}|p, s\rangle \\
&= \int d^3\vec{x} e^{-i(\vec{p}\,'-\vec{p})\cdot\vec{x}} \langle p', s'|J^0(0)|p, s\rangle \\
&= (2\pi)^3 \delta^{(3)}(\vec{p}\,' - \vec{p}) \langle p', s'|J^0(0)|p, s\rangle
\end{aligned} \tag{12}$$

In the second line, we let the momentum operators act to the left and right, and in the third, we used the integral representation of the Dirac delta function. The delta function will force $\vec{p} = \vec{p}\,'$ which means $q = 0$. Eq. (10) then gives:

$$\begin{aligned}
\langle p', s'|J^0|p, s\rangle &= \overline{u}(p', s') \left[\frac{(p' + p)^0}{2m} F_1(0) \right] u(p, s) \\
&= \frac{2E_{\vec{p}}}{2m} F_1(0) 2m\delta_{ss'} \\
&= 2E_{\vec{p}} F_1(0) \delta_{ss'}
\end{aligned}$$

where $E_{\vec{p}} = p^0 = p'^0$. Plugging this back into our expectation value for the charge and simplifying:

$$\langle p', s'|\hat{Q}|p, s\rangle = (2\pi)^3\delta^{(3)}(\vec{p}' - \vec{p})2E_{\vec{p}}F_1(0)\delta_{ss'}$$

Comparing to eq. (11) we conclude that $F_1(0) = 1$ to all orders in perturbation theory. Therefore, we conclude that the magnetic moment of a fermion is shifted by $1 + F_2(0)$ and must work now on finding $F_2(0)$.

2.2.2 *Determining* $F_2(0)$

To get a_μ, we will evaluate $\langle p', s'|J^\mu|p, s\rangle$ and pick out the part that corresponds to $F_2(0)$. There are four diagrams we need to consider, shown in Fig. 1 with the original tree diagram. We can

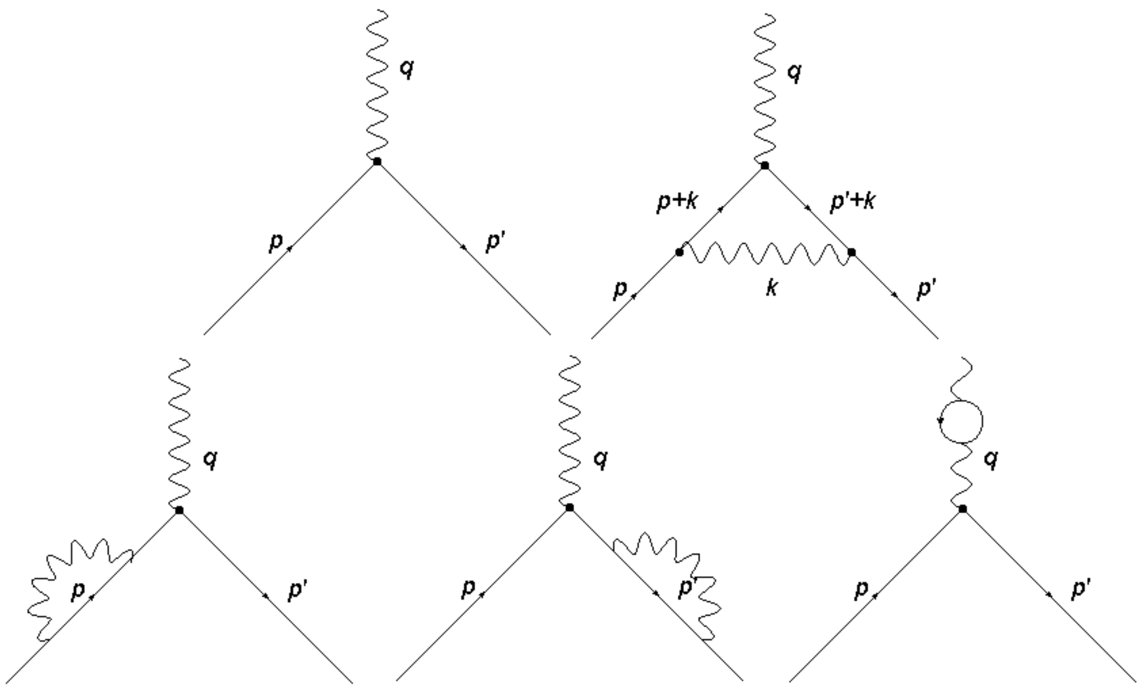


Figure 1: Tree diagram and one loop photon diagrams.

return to, eq. (9), and use the Gordon Decomposition again on the second term instead of the first to get:

$$\langle p', s'|J^\mu|p, s\rangle = \overline{u}(p', s')\left[\gamma^\mu[F_1(0) + F_2(0)] - \frac{(p' + p)^\mu}{2m}F_2(0)\right]u(p, s) \qquad (13)$$

From this, we see that to isolate $F_2(0)$ we can throw away any terms proportional to γ^μ and keep only terms that are proportional to p'^μ and p^μ. The bottom three diagrams all contain an γ^μ that is not sandwiched between any other gamma matrices. However, the second diagram on the top will have a γ^μ that is sandwiched between two other gamma matrices. Therefore, it will be the only diagram that contributes to $F_2(0)$.

We can normalize the contribution from this diagram by summing it with the contribution from the tree diagram, as we did with eq. (8), so that we get $\overline{u}(\gamma^\mu + \Gamma^\mu)u$ where Γ^μ is the contribution from the second diagram.

Applying the Feynman rules (Appendix C), we get:

$$\Gamma^\mu = \int \frac{d^4k}{(2\pi)^4} ie\gamma^\nu \frac{i}{\not{p}' + \not{k} - m} ie\gamma^\mu \frac{i}{\not{p} + \not{k} - m} ie\gamma_\nu \frac{-i}{k^2} \tag{14}$$

where m is the mass of the fermion. We also use Feynman's slash notation here where $\not{p} = p_\mu \gamma^\mu$. Our equation can be rewritten as:

$$\Gamma^\mu = -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \frac{\not{p}' + \not{k} + m}{(p' + k)^2 - m^2} \gamma^\mu \frac{\not{p} + \not{k} + m}{(p + k)^2 - m^2} \gamma_\nu \frac{1}{k^2} \tag{15}$$

which we will write as:

$$\Gamma^\mu = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu}{D}$$

with:

$$N^\mu = \gamma^\nu (\not{p}' + \not{k} + m) \gamma^\mu (\not{p} + \not{k} + m) \gamma_\nu \tag{16}$$

and:

$$D = [(p' + k)^2 - m^2][(p + k)^2 - m^2][k^2] \tag{17}$$

Now, we can combine these denominator factors using one of Feynman's identities:

$$\frac{1}{xyz} = 2 \int \int d\alpha d\beta \frac{1}{[z + \alpha(x - z) + \beta(y - z)]}$$

where the integral is over a triangle bounded by $\alpha = 0, \beta = 0, \alpha + \beta = 1$. Thus:

$$\frac{1}{D} = 2 \int d\alpha d\beta \frac{1}{\mathcal{D}} \tag{18}$$

where

$$\frac{1}{\mathcal{D}} = \frac{1}{[k^2 + 2k(\alpha p' + \beta p)]^3} \tag{19}$$

Next, we complete the square and let $\ell = k + (\alpha p' + \beta p)$ so that:

$$\frac{1}{\mathcal{D}} = \frac{1}{[\ell^2 - (\alpha + \beta)^2 m^2]^3} \tag{20}$$

Returning to eq. (16) and substituting in ℓ for k as we did for $\mathcal{D}$, we have:

$$N^\mu = \gamma^\nu[\not{\ell} + \not{P}' + m]\gamma^\mu[\not{\ell} + \not{P} + m]\gamma_\nu \tag{21}$$

where $\not{P}' = \not{p}'(1-\alpha) - \beta\not{p}$ and $\not{P} = \not{p}(1-\beta) - \alpha\not{p}'$. It will easiest to organize the terms according to their power of m. Multiplying out everything, we have:

$$\begin{aligned} N^\mu = m^2\gamma^\nu\gamma^\mu\gamma_\nu + m(\gamma^\nu\not{\ell}\gamma^\mu\gamma_\nu + \gamma^\nu\not{P}'\gamma^\mu\gamma_\nu + \gamma^\nu\gamma^\mu\not{\ell}\gamma_\nu + \gamma^\nu\gamma^\mu\not{P}\gamma_\nu) \\ + \gamma^\nu\not{\ell}\gamma^\mu\not{\ell}\gamma_\nu + \gamma^\nu\not{\ell}\gamma^\mu\not{P}\gamma_\nu + \gamma^\nu\not{P}'\gamma^\mu\not{\ell}\gamma_\nu + \gamma^\nu\not{P}'\gamma^\mu\not{P}\gamma_\nu \end{aligned} \tag{22}$$

2.2.3 m^2 *term*

A "contraction theorem" (listed in Appendix D) says:

$$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu \tag{23}$$

Therefore, the m^2 term is a γ^μ term and can be ignored.

2.2.4 m *term*

The terms proportional to m from eq. (22) are:

$$m(\gamma^\nu\not{\ell}\gamma^\mu\gamma_\nu + \gamma^\nu\not{P}'\gamma^\mu\gamma_\nu + \gamma^\nu\gamma^\mu\not{\ell}\gamma_\nu + \gamma^\nu\gamma^\mu\not{P}\gamma_\nu) \tag{24}$$

Consider the ℓ terms first. Since $k = \ell - (\alpha p' + \beta p)$ we have $d^4\ell = d^4k$. Thus, our original integral over k is now over ℓ, and the integrals all still run from $-\infty$ to ∞. Thus, any term that is linear in ℓ should integrate zero. Therefore, we can drop any term linear in ℓ leaving:

$$\begin{aligned} m(\gamma^\nu\not{P}'\gamma^\mu\gamma_\nu + \gamma^\nu\gamma^\mu\not{P}\gamma_\nu) &= m(4P'^\mu + 4P^\mu) \\ &= 4m\left[p'^\mu(1-\alpha) - \beta p^\mu + p^\mu(1-\beta) - \alpha p'^\mu\right] \end{aligned}$$

In the first line we made use of the contraction theorem (Appendix D):

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda}$$

Now, consider the integral over α and β. The rest of the integrand, and the integration region, are symmetric under renaming $\alpha \leftrightarrow \beta$. Therefore, we can symmetrize the above under an exchange of α and β to obtain:

$$= 4m(1-\alpha-\beta)(p'+p)^\mu$$

2.2.5 m^0 *term*

From eq. (22), the m^0 terms are:

$$\gamma^\nu \not{\ell} \gamma^\mu \not{\ell} \gamma_\nu + \gamma^\nu \not{\ell} \gamma^\mu \not{P} \gamma_\nu + \gamma^\nu \not{P}' \gamma^\mu \not{\ell} \gamma_\nu + \gamma^\nu \not{P}' \gamma^\mu \not{P} \gamma_\nu \tag{25}$$

Consider the l^2 term first:

$$\begin{aligned} &= \gamma^\nu \not{\ell} \gamma^\mu \not{\ell} \gamma_\nu \\ &= \ell_\lambda \ell_\sigma [\gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\sigma \gamma_\nu] \end{aligned}$$

If we remember that this appears under an integral over ℓ, we can use a handy trick to simplify this expression. The integral has the general form:

$$\int d^4\ell \; \ell_\mu \ell_\nu f(\ell^2)$$

By Lorentz invariance, this must transform like a 2-index tensor. The only available tensor is the Minkowski metric. Thus:

$$\int d^4\ell \; \ell_\mu \ell_\nu f(\ell^2) = g_{\mu\nu} C$$

where C is some constant that we will determine here. If we contract both sides by $g^{\mu\nu}$, we obtain:

$$\begin{aligned} g^{\mu\nu} \int d^4\ell \; \ell_\mu \ell_\nu f(\ell^2) &= g^{\mu\nu} g_{\mu\nu} C \\ \int d^4\ell \; \ell^2 f(\ell^2) &= 4C \\ \frac{1}{4} \int d^4\ell \; \ell^2 f(\ell^2) &= C \end{aligned}$$

And thus:

$$\int d^4\ell \; \ell_\mu \ell_\nu f(\ell^2) = g_{\mu\nu} \frac{1}{4} \int d^4\ell \; \ell^2 f(\ell^2)$$

The net effect is that, under the integral we can replace:

$$\ell_\mu \ell_\nu \to \frac{1}{4} g_{\mu\nu} \ell^2$$

Substituting this result into our equation above, our ℓ^2 term becomes:

$$\begin{aligned}\ell_\lambda \ell_\sigma[\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma_\nu] &= \frac{1}{4}g_{\lambda\sigma}\ell^2\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma_\nu \\ &= \frac{1}{4}\ell^2(4\gamma^\mu)\end{aligned}$$

To get to the last expression, we first used a contraction theorem, then reordered the Minkowski metric, and then used one final contraction theorem. Thus, only γ^μ remains and we can throw this term out. For the same reasons as before, we can throw out the terms linear in ℓ as they will go to zero under the integral. This leaves us with, from eq. (25):

$$\begin{aligned}\gamma^\nu \not{P}'\gamma^\mu\not{P}\gamma_\nu &= -2\not{P}\gamma^\mu\not{P}' \\ &= -2[(1-\beta)\not{p} - \alpha\not{p}']\gamma^\mu[(1-\alpha)\not{p}' - \beta\not{p}]\end{aligned}$$

The Dirac equation implies that $(\not{p} - m)u(p,s) = 0$ and $\overline{u}(\not{p} - m) = 0$. Since Γ^μ is sandwiched between $\overline{u}(p', s')$ and $u(p,s)$, we can replace $\not{p}$ acting to the right with m, and the same for $\not{p}'$ acting to the left. So this contribution reduces to:

$$-2[(1-\beta)\not{p} - \alpha m]\gamma^\mu[(1-\alpha)\not{p}' - \beta m]$$

Now, we will again separate out our terms in powers of m:

$$-2[(1-\beta)\not{p}\gamma^\mu(1-\alpha)\not{p}' - m[(1-\beta)\not{p}\gamma^\mu\beta + \alpha\gamma^\mu(1-\alpha)\not{p}'] + \alpha\beta m^2\gamma^\mu] \tag{26}$$

Again, the m^2 is proportional to γ^μ so we can throw it away. The m term from eq. (26) is:

$$2m[\beta(1-\beta)\not{p}\gamma^\mu + \alpha\gamma^\mu(1-\alpha)\not{p}']$$

We can again symmetrize under $\alpha \leftrightarrow \beta$ to obtain the equivalent form:

$$2m\left[\frac{1}{2}(\not{p}\gamma^\mu + \gamma^\mu\not{p}')\right][\beta(1-\beta) + \alpha(1-\alpha)] = m[2p^\mu - \gamma^\mu\not{p} + 2p'^\mu - \not{p}'\gamma^\mu][\beta(1-\beta) + \alpha(1-\alpha)]$$

Next, we use the fact that our terms are sandwiched between $\overline{u}(p', s')$ and $u(p,s)$ to get:

$$m[2p^\mu - \gamma^\mu m + 2p'^\mu - m\gamma^\mu][\beta(1-\beta) + \alpha(1-\alpha)]$$

The two γ^μ's can be thrown away, and the final result for the m term is:

$$2m(p'+p)^\mu[\beta(1-\beta)+\alpha(1-\alpha)]$$

Finally, we consider the m^0 term:

$$-2(1-\beta)(1-\alpha)\not{p}\gamma^\mu\not{p}'$$

We can simplify this using the anticommutation relation for the gamma matrices. We can complete this step multiple times and drop any γ^μ terms along the way to find:

$$\begin{aligned}-2(1-\beta)(1-\alpha)[2p^\mu\not{p}'-\gamma^\mu\not{p}\not{p}'] &= -2(1-\beta)(1-\alpha)[2p^\mu\not{p}'-2p'p\gamma^\mu+\gamma^\mu\not{p}'\not{p}] \\ &= -2(1-\beta)(1-\alpha)[2p^\mu\not{p}'+2p'^\mu\not{p}-\not{p}'\gamma^\mu\not{p}]\end{aligned}$$

Again, these are sandwiched between our $\overline{u}$ and u spinors so using that fact and dropping another γ^μ term, we get the final result:

$$-4m(p'+p)^\mu(1-\beta)(1-\alpha)$$

Putting all the terms together, the terms proportional to $(p'+p)^\mu$ are:

$$\begin{aligned}N^\mu = 4m(1-\alpha-\beta)(p'+p)^\mu + 2m(p'+p)^\mu[\beta(1-\beta)+\alpha(1-\alpha)] - 4m(p'+p)^\mu(1-\beta)(1-\alpha) \\ = 2m(p'+p)^\mu(\alpha+\beta-\alpha^2-\beta^2-2\alpha\beta) \\ = 2m(p'+p)^\mu(\alpha+\beta)(1-\alpha-\beta) \qquad (27)\end{aligned}$$

And now, we can do the integral and evaluate $F_2(0)$. Using eq. (20) and eq. (27), we have:

$$\Gamma^\mu = -2ie^2\int d\alpha\int d\beta\int\frac{d^4\ell}{(2\pi)^4}\frac{2m(p'+p)^\mu(\alpha+\beta)(1-\alpha-\beta)}{[\ell^2-(\alpha+\beta)^2m^2]^3} \qquad (28)$$

We can do the integral over ℓ by using the basic formula for Feynman integrals:

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2]^3} = \frac{-i}{32\pi^2m^2}$$

Thus:

$$\Gamma^\mu = -2ie^2\int d\alpha\int d\beta\;\; 2m(p'+p)^\mu(\alpha+\beta)(1-\alpha-\beta)\frac{-i}{32\pi^2(\alpha+\beta)^2m^2} \qquad (29)$$

Finally, we can simplify eq. (29) and do the integral over α and β remembering our integral is over a triangle formed by $\alpha = 0, \beta = 0, \alpha + \beta = 1$. The result is:

$$\Gamma^\mu = -\frac{e^2}{8\pi^2}\frac{(p' + p)^\mu}{2m} \tag{30}$$

Comparing this to eq. (13), we see that:

$$F_2(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi} \tag{31}$$

with the fine structure constant, $\alpha = e^2/4\pi \approx 1/137$. This is the correction to the magnetic moment obtained first by Schwinger in 1948. Its numerical value is:

$$a_{QED} = \frac{g-2}{2} = 0.00116$$

This calculation shows the basic techniques used to evaluate the contributions of other particles and interactions to a_μ.

3 Weak Interaction Corrections

The electroweak theory is a unified theory of the electromagnetic and weak interactions, for which Sheldon Lee Glashow, Steven Weinberg, and Abdus Salam shared the 1972 Nobel Prize. The theory became fully realized with the discovery of the Higgs boson in 2012 at the Large Hadron Collider (LHC). These ideas are important in many contexts, including various attempts to go beyond the Standard Model. Therefore, I will review them here.

Both QED and the electroweak theory are gauge theories. Their interactions are determined by a special "symmetry" known as gauge symmetry, and are conveniently expressed by specifying a certain mathematical group. For QED, the group is U(1), while for the electroweak theory, the group is the product group SU(2)$\times$U(1). The symmetry here is essential for the mathematical consistency of the theory.

A key feature of the weak interactions is that the gauge bosons that mediate the weak interaction are massive, unlike the photon. Problems arise when trying to describe the weak interaction as a gauge theory since the gauge invariance requires that the bosons be strictly massless. In the late 1960's and early 1970's, this problem was resolved by allowing the gauge symmetry to be "spontaneously broken." This idea refers to a situation in which the ground state (vacuum) of the theory does not possess the full symmetry of the equations of motion. An common example of this is a ferromagnet, which can be thought of as an array of spins with dipole-dipole interactions. These interactions are invariant under spatial rotations. However, in the ground state, all the spins are aligned in one particular direction, which minimizes the energy of the system. Therefore, the ground state spontaneously breaks the rotational invariance. In this case, we are left with a remnant symmetry, invariance under rotations about the magnetization direction. In terms of group theory, we say that the SO(3) invariance has been broken down to SO(2)(or U(1)).

A profound result regarding these spontaneously broken symmetries is the appearance of Goldstone bosons, massless excitations with particular interactions. In the ferromagnet, these would be very long wavelength "spin waves," excitations where the spins gradually deviate from the magnetization direction over arbitrarily long distances. Since the wavelengths can be arbitrarily long, the energy can be arbitrarily low. This is the characteristic of a massless excitation.

Now, when the symmetry that is broken is a gauge symmetry, something very different occurs. In this case, the would-be Goldstone boson is "eaten" by the original massless gauge field, resulting in a massive gauge field. This is known as the Higgs mechanism and is the only way to reconcile local gauge invariance with massive gauge particles.

So in the electroweak theory, the full gauge symmetry is broken by a Higgs field down to a remnant, the U(1) of QED. The gauge bosons associated with the broken symmetries get masses and are the $W^{\pm}$ and Z bosons. The interactions are tightly constrained and there is no freedom once the symmetry is specified. This has all been confirmed experimentally over the past 40+ years in hundreds of high-precision tests. The culmination of all this work was the discovery of the Higgs particle itself, with all of the expected properties.

These ideas are important for our concerns as they may play a role in possible extensions of the Standard Model. For example, the strong, weak, and EM couplings approximately unify at a high energy scale, suggesting the Standard Model may itself be the result of spontaneous breaking of a "grand unified" theory. This GUT would be based on a bigger gauge group, whose spontaneous breakdown leaves additional massive gauge bosons whose effects we could try to measure. However, these additional bosons would be much too massive to produce in present or future accelerators. The most dramatic of these bosons would be one that mediates proton decay, something yet to be observed. More importantly for us, these super-heavy gauge bosons would also make minute corrections to a_μ and other observables. Therefore, it is of great interest to make very precise calculations and measurements to see if these effects can be discovered.

One of the most prevalent theories that goes beyond the Standard Model is supersymmetry(SUSY). SUSY is a proposed symmetry that relates fermions and bosons. If true, it would imply that each particle in the Standard Model has a "superpartner" with a different spin; every boson would have a fermion-like superpartner and vice versa. The fact that these particles have not yet been discovered is a sign that SUSY, if true, is itself a broken symmetry, possibly spontaneously broken. The LHC's primary focus currently is searching for SUSY.

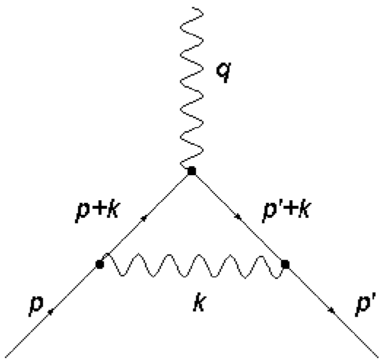
For our purposes, the main feature of SUSY is that, if true, there will be many new particles with spin $0, \frac{1}{2}, 1$. These will also make contributions to a_μ for the muon or electron. Again, comparing precision calculations and experiments may provide indirect evidence of these SUSY particles and give various clues to the nature of new physics.

As a template for a_μ calculations in a GUT or in SUSY, we can examine the electroweak theory itself. As a spontaneously broken gauge theory, it is structurally very similar to GUTs, differing only in technical details like couplings and group theory factors. In addition, the EW calculations are most conveniently carried out in a framework where one incorporates the Goldstone modes explicitly, i.e., includes them as separate particles. This is just a technicality, however, as one could always absorb the Goldstone modes into the massive gauge fields and work with those directly. The main point is that the Goldstone bosons are spin 0 particles, and so again, the EW calculation contains the ingredients needed for SUSY, where spin 0 particles are more common than spin $1/2$.

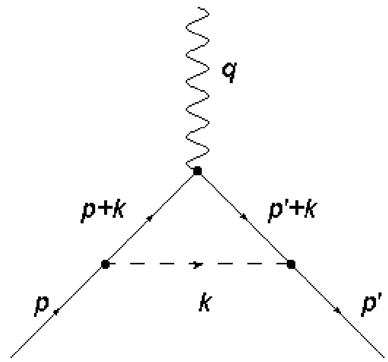
We will therefore carry out the EW calculation in full detail, with the idea of relating it to GUT or SUSY models later.

3.1 Z Boson Contribution

There are two diagrams that will contribute to a_μ here. These diagrams are shown in Fig. 2.



(a) Z boson one loop diagram.



(b) Z Goldstone boson one loop diagram.

Figure 2: One loop diagrams for the Z boson.

We will calculate the contribution from the left diagram first. The Feynman rules (Appendix C) now give:

$$\Gamma^\mu = \frac{-ig^2}{16\cos^2\theta_w}\int \frac{d^4k}{2\pi^4}\frac{N^\mu}{D} \tag{32}$$

where:

$$N^\mu = \gamma^\nu[(-1+4\sin^2\theta_w)-\gamma^5](\not{p}'+\not{k}-m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu[(-1+4\sin^2\theta_w)-\gamma^5]$$
$$\frac{1}{D} = \frac{1}{(p'+k)^2-m^2}\frac{1}{(p+k)^2-m^2}\frac{1}{k^2-M_z^2}$$

We again use Feynman's identity to combine denominators:

$$\frac{1}{D} = 2\int d\alpha d\beta \frac{1}{[(k+(\alpha p'+\beta p))^2 - m^2(\alpha+\beta)^2 + M_z^2(\alpha+\beta-1)]^3}$$

As before, we let $\ell = k + (\alpha p' + \beta p)$ to obtain:

$$\frac{1}{D} = 2\int d\alpha d\beta \frac{1}{[\ell^2 - m^2(\alpha+\beta)^2 + M_z^2(\alpha+\beta-1)]^3} \tag{33}$$

Our job is again to extract from N^μ the terms proportional to $(p'+p)^\mu$. We can separate N^μ into four different terms that we will consider separately:

$$\begin{aligned}
T_1 &= \gamma^\nu(1-4\sin^2\theta_w)(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu(1-4\sin^2\theta_w) \\
T_2 &= \gamma^\nu\gamma^5(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu(1-4\sin^2\theta_w) \\
T_3 &= \gamma^\nu(1-4\sin^2\theta_w)(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu\gamma^5 \\
T_4 &= \gamma^\nu\gamma^5(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu\gamma^5
\end{aligned}$$

We can move around the γ^5 terms to try and simplify our expression further using the identities:

$$\gamma^5\gamma^5 = 1 \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

which follows from our definition of γ^5. After this, we can combine the terms with only one γ^5 in them which will eliminate any of their m^1 terms. Therefore, we are left with three terms:

$$T_1 = (1-4\sin^2\theta_w)^2\gamma^\nu(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu \tag{34}$$

$$T_2+T_3 = 2(1-4\sin^2\theta_w)[\gamma^\nu\gamma^5(\not{p}'+\not{k})\gamma^\mu(\not{p}+\not{k})\gamma_\nu + m^2\gamma^\nu\gamma^5\gamma^\mu\gamma_\nu] \tag{35}$$

$$T_4 = \gamma^\nu(\not{p}'+\not{k}-m)\gamma^\mu(\not{p}+\not{k}-m)\gamma_\nu \tag{36}$$

We can now begin to simplify each term in much the same way as we did with the QED calculation. For eq. (34), we again let:

$$\not{\ell} = \not{k} - (\alpha\not{p}'+\beta\not{p}) \qquad \not{P}' = (1-\alpha)\not{p}'-\beta\not{p} \qquad \not{P} = (1-\beta)\not{p}-\alpha\not{p}$$

to obtain:

$$T_1 = (1-4\sin^2\theta_w)^2\gamma^\nu(\not{\ell}+\not{P}'+m)\gamma^\mu(\not{\ell}+\not{P}+m)\gamma_\nu$$

But if we look back at our previous calculation for QED, it is easy to see that this is just the same term from that calculation multiplied by $(1-4\sin^2\theta_w)^2$. Hence:

$$T_1 = 2m(p'+p)^\mu(\alpha+\beta)(1-\alpha-\beta)(1-4\sin^2\theta_w)^2 \tag{37}$$

We will next consider eq. (36) since it does not include any γ^5 terms. Again, we use our definitions of ℓ, P', and P to get:

$$
\begin{aligned}
T_4 =&\gamma^\nu(\not{\ell}+\not{P}'-m)\gamma^\mu(\not{\ell}+\not{P}-m)\gamma_\nu \\
=&\gamma^\nu\not{\ell}\gamma^\mu\not{\ell}\gamma_\nu+\gamma^\nu\not{\ell}\gamma^\mu\not{P}\gamma_\nu+\gamma^\nu\not{P}'\gamma^\mu\not{\ell}\gamma_\nu+\gamma^\nu\not{P}'\gamma^\mu\not{P}\gamma_\nu \\
&-m(\gamma^\nu\not{\ell}\gamma^\mu\gamma_\nu+\gamma^\nu\not{P}'\gamma^\mu\gamma_\nu+\gamma^\nu\gamma^\mu\not{\ell}\gamma_\nu+\gamma^\nu\gamma^\mu\not{P}\gamma_\nu) \\
&+m^2\gamma^\nu\gamma^\mu\gamma_\nu
\end{aligned}
$$

As before, we throw out any γ^μ terms and therefore drop the m^2 term. Furthermore, the first term is just our m^0 term from the QED calculation and the second term is our m term from the QED calculation with an overall minus sign. We therefore obtain:

$$
T_4 = 2m(p'+p)^\mu[(\alpha+\beta)(1-\alpha-\beta)-4(1-\alpha-\beta)] \tag{38}
$$

Finally, the term with a single γ^5 is dropped since it contributes to a different form factor not discussed previously and therefore is not relevant for the magnetic moment. The final result is thus:

$$
N^\mu = 2m(p'+p)^\mu[(\alpha+\beta)(1-\alpha-\beta)(1-4\sin^2\theta_w)^2+(\alpha+\beta)(1-\alpha-\beta)-4(1-\alpha-\beta)] \tag{39}
$$

And hence:

$$
\begin{aligned}
\Gamma^\mu = &\frac{-ig^2m}{4\cos^2\theta_w}(p'+p)^\mu\int d\alpha d\beta \\
&\int\frac{d^4l}{2\pi^4}\frac{(\alpha+\beta)(1-\alpha-\beta)(1-4\sin^2\theta_w)^2+(\alpha+\beta)(1-\alpha-\beta)-4(1-\alpha-\beta)}{\ell^2-m^2(\alpha+\beta)^2+M_z^2(\alpha+\beta-1)]^3}
\end{aligned} \tag{40}
$$

We can do the ℓ integral as before with Feynman's formula:

$$
\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-M^2]^3}=\frac{-i}{32\pi^2M^2}
$$

with:

$$
M^2 = M_Z^2(1-\alpha-\beta)+m^2(\alpha+\beta)^2
$$

Now, for the muon:

$$
\frac{m}{M_Z}=\frac{0.106 \text{ GeV}}{91 \text{ GeV}}\sim 10^{-3}
$$

Therefore, an expansion in powers of m/M_Z makes sense. To leading order, this amounts to setting $m = 0$ in M^2, which results in:

$$\Gamma^\mu = \frac{-g^2 m}{128\pi^2 \cos^2\theta_w}(p'+p)^\mu \int d\alpha d\beta \\ \frac{(\alpha+\beta)(1-\alpha-\beta)(1-4\sin^2\theta_w)^2 + (\alpha+\beta)(1-\alpha-\beta) - 4(1-\alpha-\beta)}{M_Z^2(1-\alpha-\beta)}$$

The first correction to this will be of order $m/M_Z^2 \sim 10^{-5}$. Therefore, after canceling the $(1-\alpha-\beta)$ terms, we have:

$$\Gamma^\mu = \frac{-g^2 m}{128\pi^2 M_Z^2 \cos^2\theta_w}(p'+p)^\mu \int d\alpha d\beta\, (\alpha+\beta)(1-4\sin^2\theta_w)^2 + (\alpha+\beta) - 4$$

Finally, we can evaluate the remaining integrals:

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta(\alpha+\beta)(1-4\sin^2\theta_w)^2 + (\alpha+\beta) - 4 = \left[-\frac{4}{3} - \frac{8}{3}\sin^2\theta_w + \frac{16}{3}\sin^4\theta_w\right]$$

to obtain:

$$\Gamma^\mu = -\frac{G_F m^2}{8\sqrt{2}\pi^2}\frac{(p'+p)^\mu}{2m}\left[-\frac{4}{3} - \frac{8}{3}\sin^2\theta_w + \frac{16}{3}\sin^4\theta_w\right] \tag{41}$$

In this result, we have used the fact that $M_Z^2\cos^2\theta_w = M_W^2$ and introduced the Fermi constant:

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2}$$

The contribution, $F_2(0)$, to the anomaly is as before the coefficient of $-(p'+p)^\mu/2m$, here:

$$F_2(0) = \frac{G_F m^2}{8\sqrt{2}\pi^2}\left[-\frac{4}{3} - \frac{8}{3}\sin^2\theta_w + \frac{16}{3}\sin^4\theta_w\right] \tag{42}$$

Next, we consider the diagram from Fig. 2 that inserts the Goldstone boson. The Feynman rules give:

$$\Gamma^\mu = \frac{-ig_z^2 m^2}{4M_W^2}\int \frac{d^4k}{(2\pi)^4}\gamma^5 \frac{\not{p}' + \not{k} + m}{(p'+k)^2 - m^2}\gamma^\mu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2}\gamma^5 \frac{1}{k^2 - M_z^2} \tag{43}$$

This can be evaluated in the same way as before. However, the overal factor m^2/M_W^2 is of the same order as the term we just dropped in the Z boson contribution. (Remember that $M_Z^2\cos^2\theta_w = M_W^2$ so M_Z and M_W are of the same order.) We can and must therefore neglect the Goldstone contribution. Therefore, eq (42) gives the full Z boson contribution to a_μ.

3.2 W Boson Calculation

The W boson correction has four associated diagrams: one involving solely W bosons, two involving a W boson and a Goldstone boson, and one involving two Goldstone bosons. The diagram for only W boson corrections is shown in Fig. 3.

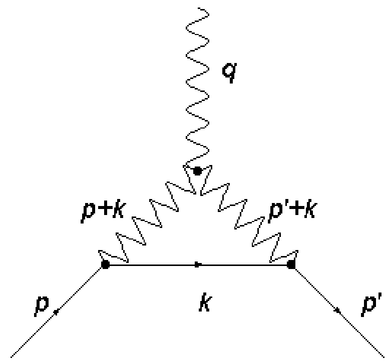


Figure 3: W boson one loop diagram.

We again write Γ^μ in its normal form:

$$\Gamma^\mu = \frac{ig^2}{8}\int \frac{d^4k}{(2\pi)^4}\frac{N^\mu}{D} \tag{44}$$

where:

$$\begin{aligned} N^\mu =& \gamma^\lambda(1-\gamma^5)\not{k}\gamma_\lambda(1-\gamma^5)(p'+p+2k)^\mu \\ & - \gamma^\mu(1-\gamma^5)\not{k}(\not{p}'+\not{k}+\not{q})(1-\gamma^5) + (\not{q}-\not{p}-\not{k})(1-\gamma^5)\not{k}\gamma^\mu(1-\gamma^5) \\ \frac{1}{D} =& \frac{1}{(p+k)^2-M_W^2}\frac{1}{(p'+k)^2-M_W^2}\frac{1}{k^2} \end{aligned}$$

We combine our denominator term again using Feynman's identity and let $\ell = k + \alpha p' + \beta p$ to find:

$$\frac{1}{D} = 2\int\int d\alpha d\beta \frac{1}{\left(\ell^2 + m^2\left[(\alpha+\beta)-(\alpha+\beta)^2 - \frac{M_W^2}{m^2}(\alpha+\beta)\right]\right)^3} \tag{45}$$

As before, we intend to find the $(p'+p)^\mu$ terms in N^μ. In our manipulation of N^μ, we drop any terms with only one γ^5. This leaves us with two terms, one with no γ^5's:

$$\gamma^\lambda\not{k}\gamma_\lambda(p'+p+2k)^\mu - \gamma^\mu\not{k}(\not{p}'+\not{k}+\not{q}) + (\not{q}-\not{p}-\not{k})\not{k}\gamma^\mu$$

and one with multiple γ^5's:

$$\gamma^\lambda\gamma^5\not{k}\gamma_\lambda\gamma^5(p'+p+2k)^\mu - \gamma^\mu\gamma^5\not{k}(\not{p}'+\not{k}+\not{q})\gamma^5 + (\not{q}-\not{p}-\not{k})\gamma^5\not{k}\gamma^\mu\gamma^5$$

If we move around the γ^5 terms in the second term, we get back the exact same thing as the first term so that with $q = p' - p$:

$$N^\mu = 2(\gamma^\lambda \not{k} \gamma_\lambda (p' + p + 2k)^\mu - \gamma^\mu \not{k}(2\not{p}\,' + \not{k} - \not{p}) - (2\not{p} + \not{k} - \not{p}')\not{k}\gamma^\mu) \tag{46}$$

We will manipulate the leading term in eq. (46) first. We reuse the contraction theorem, eq. (23), and recall $k = \ell - \alpha p' - \beta p$ to get:

$$-2(\not{\ell} - \alpha \not{p}\,' - \beta \not{p})(p' + p + 2\ell - 2\alpha p' - 2\beta p)$$

We multiply out our terms and as before any terms linear in ℓ will integrate to zero:

$$-2(2\ell^\mu \not{\ell} - m(\alpha + \beta)(1 - 2\alpha)p'^\mu - m(\alpha + \beta)(1 - 2\beta)p)$$

This expression is symmetric under interchange of α and β. Furthermore, it will have the same ℓ^2 simplification and will result in the first term being proportional to only γ^μ. These simplifications leave us with:

$$2m(p' + p)^\mu(\alpha + \beta)(1 - \alpha - \beta)$$

For the next term, we use the anticommutation relation and $k = \ell - \alpha p' - \beta p$ to obtain:

$$8p'^\mu(\not{\ell} - \alpha\not{p}' - \beta\not{p}) - 2m\gamma^\mu(\not{\ell} - \alpha\not{p}' - \beta\not{p})$$

As before, we drop any terms proportional to ℓ or γ^μ and use Dirac's equation to replace any $\not{p}'$ and $\not{p}$ with m. This results in the expression:

$$-4m(\alpha + 2\beta)p'^\mu$$

Finally, we symmetrize under α and β interchange to arrive at:

$$-6m(\alpha + \beta)p'^\mu$$

Next, we can notice that the last term is exactly the same as the second term, just with p' and p switched. Therefore, its contribution to N^μ will be the same except with p put in place of p':

$$-6m(\alpha + \beta)p^\mu$$

Hence:

$$N^\mu = 2m(p' + p)^\mu(\alpha + \beta)(2[1 - \alpha - \beta] - 3) \tag{47}$$

and:

$$\Gamma^\mu = \frac{ig^2}{4}\int d\alpha d\beta \int \frac{d^4\ell}{(2\pi)^4}\frac{2m(p'+p)^\mu(\alpha+\beta)(2[1-\alpha-\beta]-3)}{\left(\ell^2+m^2\left[(\alpha+\beta)-(\alpha+\beta)^2-\frac{M_W^2}{m^2}(\alpha+\beta)\right]\right)^3}$$

The Feynman integral formula allows to complete the $d^4\ell$ integral again:

$$\begin{aligned}\Gamma^\mu &= \frac{ig^2}{4}\int d\alpha d\beta \frac{-2mi(p'+p)^\mu(\alpha+\beta)(2[1-\alpha-\beta]-3)}{32\pi^2 m^2(\alpha+\beta)\left(-1+\alpha+\beta+\frac{M_W^2}{m^2}\right)} \\ &= \frac{g^2 m}{64\pi^2}(p'+p)^\mu \int d\alpha d\beta \frac{2(1-\alpha-\beta)-3}{M_W^2-(1-\alpha-\beta)m^2} \\ &= \frac{g^2 m^2}{32\pi^2 M_W^2}\frac{(p'+p)^\mu}{2m}\int d\alpha d\beta \frac{2(1-\alpha-\beta)-3}{1-(1-\alpha-\beta)\frac{m^2}{M_W^2}}\end{aligned}$$

We again take the leading term in an expansion in powers of m^2/M_W^2, which amounts to putting $m \to 0$ in the denominator. This gives:

$$\Gamma^\mu = \frac{G_F m^2}{4\pi^2\sqrt{2}}\frac{(p'+p)^\mu}{2m}\int d\alpha d\beta(-1-2\alpha-2\beta)$$

The integral is equal to $-7/6$ so that:

$$\Gamma^\mu = -\frac{7}{3}\frac{G_F m^2}{8\sqrt{2}\pi^2}\frac{(p'+p)^\mu}{2m}$$

and hence:

$$F_2(0) = \frac{7}{3}\frac{G_F m^2}{8\sqrt{2}\pi^2} \tag{48}$$

In addition to the W diagram we showed earlier, there are three other diagrams for the W contribution that include Goldstone bosons (Fig. 4).

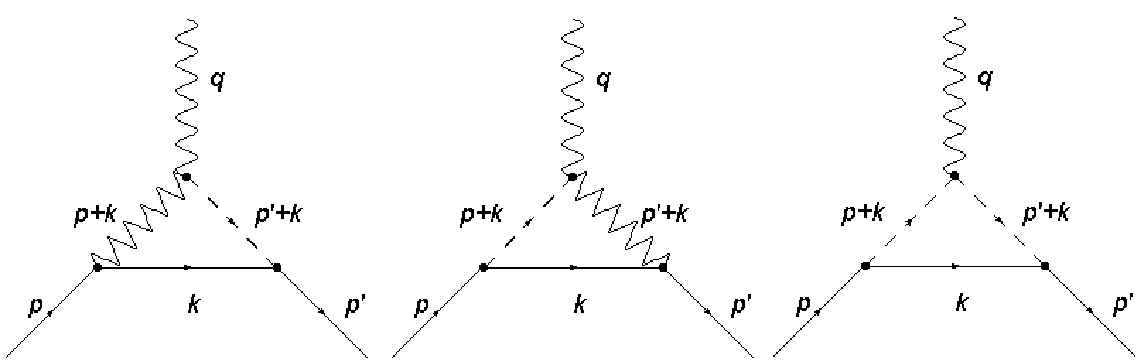


Figure 4: W Goldstone boson one loop diagrams.

For the first diagram, the Feynman rules (Appendix C) give:

$$\Gamma^\mu = \frac{ig^2}{8}\int \frac{d^4k}{(2\pi)^4}(1-\gamma^5)\not{k}\gamma^\mu(1-\gamma^5)\frac{1}{(p'+k)^2-M_W^2}\frac{1}{(p+k)^2-M_W^2}\frac{1}{k^2} \tag{49}$$

The denominator in eq. (49) is the same as the W boson so the result will just be what we have in eq. (45). In the numerator, we drop any γ^5 terms as before, leaving:

$$N^\mu = \not{k}\gamma^\mu + \gamma^5\not{k}\gamma^\mu\gamma^5$$

Rearranging the γ^5 in the second term and substituting in $k = \ell - \alpha p' - \beta p$ gives:

$$N^\mu = 2(\not{\ell} - \alpha\not{p}' - \beta\not{p})\gamma^\mu$$

We use the anticommutation relation and remove any terms proportional to ℓ or γ^μ to give the final result:

$$N^\mu = -4\beta p^\mu \tag{50}$$

We can now plug in eq. (45) and eq. (50) into eq. (49), giving:

$$\Gamma^\mu = \frac{ig^2 m}{4}\int d\alpha d\beta \int \frac{d^4\ell}{(2\pi)^4}\frac{-4\beta p^\mu}{\left(\ell^2 + m^2\left[(\alpha+\beta) - (\alpha+\beta)^2 - \frac{M_W^2}{m^2}(\alpha+\beta)\right]\right)^3}$$

Performing the momentum integration:

$$\Gamma^\mu = \frac{-g^2 m}{32\pi^2 M_W^2}\int d\alpha d\beta \frac{\beta p^\mu}{(\alpha+\beta) + \frac{m^2}{M_W^2}[(\alpha+\beta)^2 - (\alpha+\beta)]} \tag{51}$$

We will leave this for right now and work with the other diagrams first to hopefully sum them all together and make the integral simple to calculate.

The second diagram will give a correction similar to the first, with the numerator changed to:

$$N^\mu = \gamma^\mu(1-\gamma^5)\not{k}(1+\gamma^5)$$

The denominator will be exactly the same as the W boson's correction and so we can reuse that result. Multiplying everything out, dropping γ^5 terms, and substituting in for k as before, we find:

$$N^\mu = 2\gamma^\mu(\not{\ell} - \alpha\not{p}' - \beta\not{p})$$

This numerator is again similar to the first diagram, just with the γ^μ moved to the other side. This results in the same expression as eq. (50) but with $\beta \to \alpha$ and $p \to p'$:

$$N^\mu = -4\alpha p'^\mu \tag{52}$$

Thus:

$$\Gamma^\mu = -\frac{g^2 m}{32\pi^2 M_W^2} \int d\alpha d\beta \frac{\alpha p'^\mu}{(\alpha+\beta) + \frac{m^2}{M_W^2}[(\alpha+\beta)^2 - (\alpha+\beta)]} \tag{53}$$

The third diagram of Fig. 4 gives:

$$\Gamma^\mu = \frac{ig^2 m^2}{8M_W^2} \int \frac{d^4k}{(2\pi)^4} (1-\gamma^5)\not{k}(1+\gamma^5)(p'+p+2k)^\mu \frac{1}{(p'+k)^2 - M_W^2} \frac{1}{(p+k)^2 - M_W^2} \frac{1}{k^2} \tag{54}$$

We can observe, however, that with the Z Goldstone boson correction, we already have a m^2/M_W^2 term out in front. This means that this correction is subleading and can be dropped. Therefore, the full W Goldstone contribution is eq. (51) summed with eq. (53):

$$\Gamma^\mu = -\frac{g^2 m}{32\pi^2 M_W^2} \int d\alpha d\beta \frac{\alpha p' + \beta p}{(\alpha+\beta) + \frac{m^2}{M_W^2}[(\alpha+\beta)^2 - (\alpha+\beta)]}$$

We can symmetrize under interchange of α and β and introduce G_F, obtaining:

$$-\frac{G_F m^2}{4\sqrt{2}\pi^2} \frac{(p'+p)^\mu}{2m} \int d\alpha d\beta \frac{(\alpha+\beta)}{(\alpha+\beta) + \frac{m^2}{M_W^2}[(\alpha+\beta)^2 - (\alpha+\beta)]}$$

And take the leading term in m/M_W to obtain:

$$\Gamma^\mu = \frac{-G_F m^2}{8\sqrt{2}\pi^2} \frac{(p'+p)^\mu}{2m}$$

From this, we extract:

$$F_2(0) = \frac{G_F m^2}{8\sqrt{2}\pi^2} \tag{55}$$

This is the full correction from the W Goldstone boson diagrams.

The complete W Boson contribution is obtained from eqs. (48) and (55):

$$F_2(0) = \frac{10}{3} \frac{G_F m^2}{8\sqrt{2}\pi^2} \tag{56}$$

The corrections to a_μ from the electroweak theory are then complete. We can now add up all of these corrections to get an idea of the relative size of each correction separately and summed together.

4 Total Electroweak Correction to a_μ of the Muon

We first summarize the corrections appearing in eqs. (31), (43), and (56):

$$\begin{aligned}
F_2(0)_\gamma &= \frac{\alpha}{2\pi} \\
F_2(0)_Z &= \frac{G_F m^2}{8\sqrt{2}\pi^2}\left[-\frac{4}{3} - \frac{8}{3}\sin^2\theta_w + \frac{16}{3}\sin^4\theta_w\right] \\
F_2(0)_W &= \frac{10}{3}\frac{G_F m^2}{8\sqrt{2}\pi^2}
\end{aligned}$$

To get the numerical total of our corrections, we first need to get the values of our numerical constants. The fine structure constant is measured to be $\alpha^{-1} = 137.035999049$. [5]Therefore, our correction due to QED is numerically:

$$F_2(0)_\gamma = 1.16140973318 \times 10^{-3}$$

Notably, we see that the weak corrections depend on the mass of the fermion. Therefore, the muon is a more sensitive probe to a_μ since $m_\mu/m_e \sim 200$. For the Z and W boson corrections, we need to define $\sin^2\theta_w$, the Fermi constant, G_F, and the mass of the muon, m:

$$\begin{aligned}
\sin^2\theta_w = 1 - \cos^2\theta_w = 1 - \left(\frac{M_W}{M_Z}\right)^2 &= 0.2229 \\
G_F &= 1.1664 \times 10^{-5}\ \frac{1}{GeV^2} \\
m_\mu &= 0.1057\ GeV
\end{aligned}$$

With these constants defined, we can now get the numerical results for the weak corrections. For the Z boson, we conclude:

$$F_2(0)_Z = -1.941 \times 10^{-9}$$

For the W boson, we find:

$$F_2(0)_W = 3.890 \times 10^{-9}$$

These calculations show why we threw out the subleading terms in the first place. The corrections due to weak interactions are already exceptionally small compared to the corrections due to QED. If we included any of those subleading terms, their contributions would have been approximately 10^{-6} times smaller than the weak corrections.

At last, we can add together all of these corrections we have worked so hard to find, getting the final numerical result of:

$$F_2(0) = 1.16141168218 \times 10^{-3} \tag{57}$$

And so we have found the total one loop correction to a_μ of the muon from electroweak theory. The next step is now applying all that we have learned from these calculations and using it to try and bridge the gap between the current theoretical and experimental values for a_μ.

5 Testing for New Physics

We now move into our main goal of constraining the masses of supersymmetric particles, based on the discrepancy found between the current experimental and theoretical values of a_μ. The experimental result measured at Brookhaven in 2001 for a_μ is [6]:

$$a_\mu = 1\ 165\ 920\ 89(54)(63) \times 10^{-11} \tag{58}$$

Since the experiment was completed, the theoretical calculation of a_μ has improved, especially in its calculations of hadronic contributions, and the result is now:

$$a_\mu = 1\ 165\ 918\ 02(49) \times 10^{-11} \tag{59}$$

These results give a difference of:

$$\delta a_\mu = 287 \pm 80 \times 10^{-11} \tag{60}$$

This leads to a discrepancy of 3.6σ, which indicates that there may be additional contributions to a_μ beyond the Standard Model.

To show how these results can be used to constrain new physics, we will consider the case of supersymmetry. As mentioned before, SUSY relates particles of different spins and implies that every known particle has a superpartner with spin $1/2$ unit different. Therefore, fermions are the partners of bosons, and vice versa. For example, the spin 1 has a spin $1/2$ partner, the photino, while the spin $1/2$ electron has a spin 0 partner, the selectron. The naming convention is such that fermionic partners of bosons add an "-ino" suffix, while bosonic partners add an "s-" prefix. Therefore, the other fermionic partners will be the gluino (partner of the gluon), the wino and zino (partners of the W and Z), and the higgsino (partner of the Higgs boson). For the bosonic partners, we also have the spin 0 squarks (top squark, up squark, etc.) and the sneutrinos.

The motivation for SUSY is that it has the potential to solve many of the shortcomings of the Standard Model. The most important of these is the "hierarchy problem," which refers to the need for very precise fine-tuning of paramters to maintain the ratio between the energy scale of the electroweak symmetry breaking and a higher, more fundamental scale, perhaps the Planck scale. For example, the observed Higgs boson is expected to have a mass much larger than is actually observed. This could be the result of accidental fine-tuning, but it suggests that there is a symmetry instead. SUSY is able to resolve this issue because the contributions to the Higgs mass from the superpartners tends to cancel those from the Standard Model particles, resulting in much less sensitivity to physics at high energy scales.

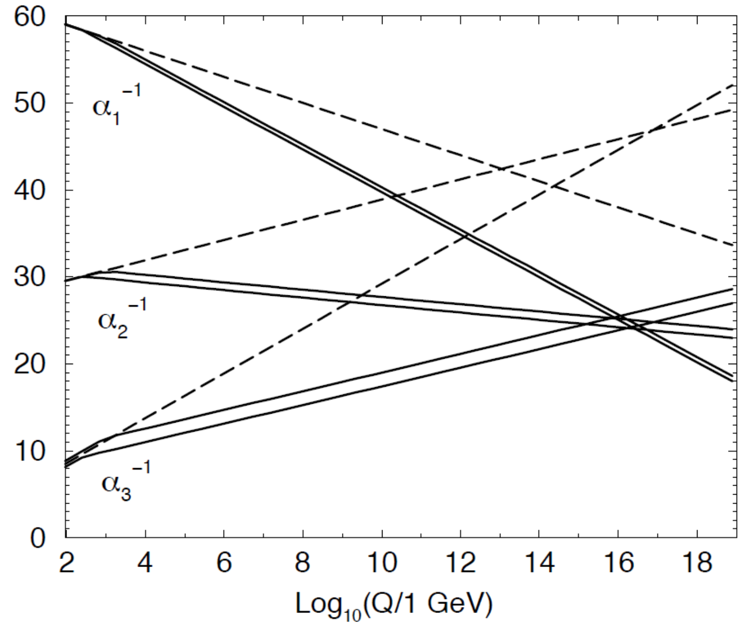


Figure 5: Unification of the couplings at high energies. Dashed lines show SM couplings, solid lines include SUSY contributions. In each case $\alpha \equiv g^2/4\pi$, and the subscripts 3, 2, 1 refer to the SU(3), SU(2)$_L$, and U(1)$_Y$ couplings, respectively.

Another suggestive hint involves the apparent unification of couplings at high scale discussed earlier in regards to the GUT. Fig. 5 shows the evolution of the Standard Model couplings with (solid line) and without SUSY. We can see that the idea of unifying them works much better when we include SUSY.

Furthermore, SUSY gives new sources of CP violation, a problem of the Standard Model, and can provide an attractive and viable candidate for dark matter. String theory also requires SUSY so if string theory correctly describes quantum gravity, then SUSY is almost certainly correct.

However, despite being the major focus of experiments at the LHC, none of the superpartners have been observed directly. The question then arises whether there existence could be detected indirectly, through something like the discrepancy in the muon a_μ results. This is what we intend to consider here.

To explore this idea, we need to evaluate the corrections to a_μ that would arise from SUSY particles. The largest corrections to a_μ will again come from one-loop diagrams. In the minimal supersymmetric standard model (MSSM), there is a coupling between the muon, smuons (scalar muons), and "neutralinos," fermionic partners to gauge bosons that are electrically neutral. Neutralinos are the collective names for the photino and zino. There is also a coupling between the muon, muon sneutrinos, and charginos, partners of the $W^\pm$. Hence, there are two new diagrams to evaluate shown in Fig. 6.

We have already done much of the work in evaluating these contributions, including calculating

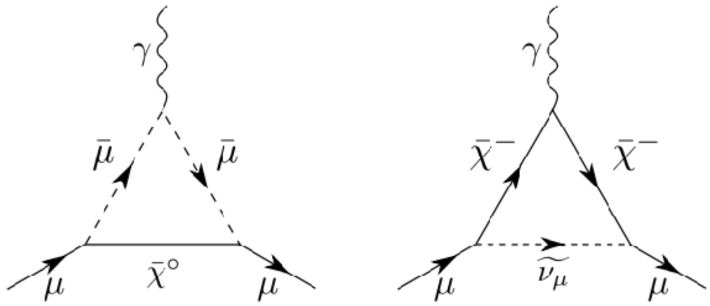


Figure 6: One-loop contributions to a_μ in minimal supersymmetry.

the contributions of gauge bosons (spin 1), fermions (spin $1/2$), and scalars (spin 0). The challenge is to include all the contributions with the correct coupling factors. Among the complexities is the fact that particles created in the interactions are generally mixtures of states with definite mass ("mass eigenstates") that appear in Feynman propagators. Therefore, there are mixing matrices that translate between the two sets of states and appear among the coefficients in the Feynman diagrams.

As an example, if we assume a MSUGRA scenario for supersymmetry breaking (see below), then in the basis of gauge eigenstates, the neutralinos have a mass matrix that can be written in the form:

$$M_{\chi^0} = \begin{pmatrix} M_1 & 0 & -m_Z\cos\beta\sin\theta_w & m_Z\sin\beta\sin\theta \\ 0 & M_2 & m_Z\cos\beta\cos\theta_w & -m_Z\sin\beta\cos\theta_w \\ -m_Z\cos\beta\sin\theta_w & m_Z\cos\beta\cos\theta_w & 0 & -\mu \\ m_Z\sin\beta\sin\theta_w & -m_Z\sin\beta\cos\theta_w & -\mu & 0 \end{pmatrix}$$

Here M_1 and M_2 are SUSY breaking “gaugino mass parameters," μ is a SUSY-respecting Higgs mass parameter, and $\tan\beta = v_u/v_d$ is the ratio of vacuum expectation values for the two Higgs fields of the theory. To find the basis of states with definite masses, we must diagonalize this matrix, i.e., find N such that:

$$N^* M_{\chi^0} N^\dagger = \mathrm{diag}(m_{\chi^0_1},\ m_{\chi^0_2},\ m_{\chi^0_3},\ m_{\chi^0_4})$$

The matrix elements of N are then part of the Feynman rules for the theory. A similar story arises for the charginos and smuons.

After incorporating all these effects, the one-loop corrections to a_μ coming from SUSY can be written as [7]:

$$\delta a_\mu^{\chi^0} = \frac{m_\mu}{16\pi^2} \sum_{i,m} \left[-\frac{m_\mu}{12 m_{\tilde{\mu}_m}^2} \left(|n_{im}^L|^2 + |n_{im}^R|^2 \right) F_1^N(x_{im}) + \frac{m_{\chi_i^0}}{3 m_{\tilde{\mu}_m}^2} Re\left(n_{im}^L n_{im}^R \right) F_2^N(x_{im}) \right] \quad (61)$$

and:

$$\delta a_\mu^{\chi^\pm} = \frac{m_\mu}{16\pi^2} \sum_{k} \left[-\frac{m_\mu}{12 m_{\tilde{\nu}_\mu}^2} \left(|c_k^L|^2 + |c_k^R|^2 \right) F_1^C(x_k) + \frac{m_{\chi_k^\pm}}{3 m_{\tilde{\nu}_\mu}^2} Re\left(c_k^L c_k^R \right) F_2^C(x_k) \right] \quad (62)$$

where $i = 1, 2, 3, 4$, $m = 1, 2$, and $k = 1, 2$ are the neutralino, smuon, and chargino mass eigenstate labels respectively. The kinematic loop functions are defined as:

$$\begin{aligned}
F_1^N(x) &= \frac{2}{(1-x)^4} \left[1 - 6x + 3x^2 + 2x^3 - 6x^2 \ln x \right] \\
F_2^N(x) &= \frac{3}{(1-x)^3} \left[1 - x^2 + 2x \ln x \right] \\
F_1^C(x) &= \frac{2}{(1-x)^4} \left[2 + 3x - 6x^2 + x^3 + 6x \ln x \right] \\
F_2^C(x) &= -\frac{3}{2(1-x)^3} \left[3 - 4x + x^2 + 2 \ln x \right]
\end{aligned}$$

They are normalized such that $F_1^N(1) = F_2^N(1) = F_1^C(1) = F_2^C(1)$. These functions depend on the variables x_{im} and x_k, defined as:

$$x_{im} = \frac{m_{\chi_{im}^0}^2}{m_{\tilde{\mu}_m}^2} \qquad x_k = \frac{m_{\chi_k^\pm}^2}{m_{\tilde{\nu}_\mu}^2}$$

where $m_{\chi_{im}^0}^2$, $m_{\tilde{\mu}_m}^2$, $m_{\chi_k^\pm}^2$, and $m_{\tilde{\nu}_\mu}^2$ are the mass eigenstates of the neutralino, smuon, chargino, and muon sneutrino respectively.

We further define:

$$\begin{aligned}
n_{im}^R &= \sqrt{2} g_1 N_{i1} X_{m2} + y_\mu N_{i3} X_{m1} \\
n_{im}^L &= \frac{1}{\sqrt{2}} \left(g_2 N_{i2} + g_1 N_{i1} \right) X_{m1}^* - y_\mu N_{i3} X_{m2}^* \\
c_k^R &= y_\mu U_{k2} \\
c_k^L &= -g_2 V_{k1}
\end{aligned}$$

where the $*$ indicates the complex conjugate. The parameters $g_2 \simeq 0.66$ and $g_1 \simeq 0.36$ are the $SU(2)$ and $U(1)$ gauge couplings. Furthermore, $y_\mu = g_2 m_\mu / \sqrt{2} m_W \cos\beta$ is the muon Yukawa coupling. The above expressions also contain the neutralino (N_{ij}), chargino (U_{kl} and V_{kl}), and smuon (X_{mn}) mass mixing matrices, which satisfy:

$$\begin{aligned} N^* M_{\chi^0} N^\dagger &= diag(m_{\chi_1^0},\ m_{\chi_2^0},\ m_{\chi_3^0},\ m_{\chi_4^0}) \\ U^* M_{\chi^\pm} V^\dagger &= diag(m_{\chi_1^\pm},\ m_{\chi_2^\pm}) \\ X M_{\widetilde{\mu}}^2 X^\dagger &= diag(m_{\widetilde{\mu}_1}^2,\ m_{\widetilde{\mu}_2}^2) \end{aligned}$$

where each mass shown in the diagonalized matrix is a mass eigenstate of the neutralino, chargino, and smuon respectively. The mass matrices for the chargino and smuon are:

$$M_{\chi^\pm} = \begin{pmatrix} M_2 & \sqrt{2} m_W \sin\beta \\ \sqrt{2} m_W \cos\beta & \mu \end{pmatrix}$$

and:

$$M_{\widetilde{\mu}}^2 = \begin{pmatrix} m_L^2 + \left(\sin^2\theta_w - \frac{1}{2}\right) m_Z^2 \cos 2\beta & m_\mu \left(A_{\widetilde{\mu}}^* - \mu \tan\beta\right) \\ m_\mu \left(A_{\widetilde{\mu}} - \mu^* \tan\beta\right) & m_R^2 - m_Z^2 \sin^2\theta_w \cos 2\beta \end{pmatrix}$$

These matrices, along with the neutralino mass matrix from earlier, determine our phase convention for the parameter μ. ($A_{\widetilde{\mu}}$ is a constant parameter here, not our electromagnetic potential from earlier.) Furthermore, θ_w is the usual weak mixing angle that we defined earlier. We can now

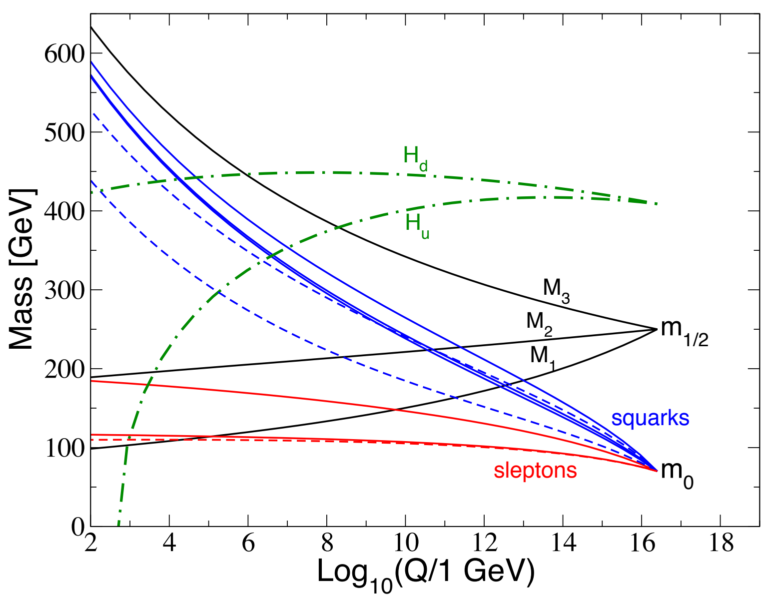


Figure 7: Unification of mass parameters at high energies in a MSUGRA model.

study these results for various values of the parameters that appear, in order to see what would be consistent with the experimental value of δa_μ. As a typical example, we can consider the "minimal supergravity" (MSUGRA) scenario for SUSY breaking. This makes specific assumptions about the superpartner masses and their couplings, with the result that, at high energies (where our theories all unify and become simpler), there are only three independent mass parameters, known as $m_{1/2}$, m_0, and A_0. $m_{1/2}$ is a gaugino mass parameter for which M_1, M_2, and M_3 all converge at unification. m_0 is a scalar mass parameter for squarks and sleptons, and A_0 parameterizes a three-point coupling between scalars. From inputting these values with $\tan\beta$, we can calculate all other masses and couplings.

In Fig. 7, we show a typical expectation for theories like MSUGRA. The horizontal axis shows the energy scale while the vertical axis is our mass scale. We see that at low energies, the pattern of masses looks fairly complicated, but that at high energies, these masses converge and the model simplifies considerably.

To present some illustrative calculations, we used the programs SOFTSUSY [8] and Supermodel ([9]) to determine the couplings. SOFTSUSY takes the MSUGRA inputs for each parameter and Supermodel translates these into the couplings and masses that are needed. We can then assemble the results from eqs. (61) and (62).

In Fig. 8, we show the resulting SUSY contributions to a_μ as a function of the lightest chargino and smuon masses, and for three different values of $\tan\beta$. The $\pm 1\sigma$ and $\pm 2\sigma$ bounds are also

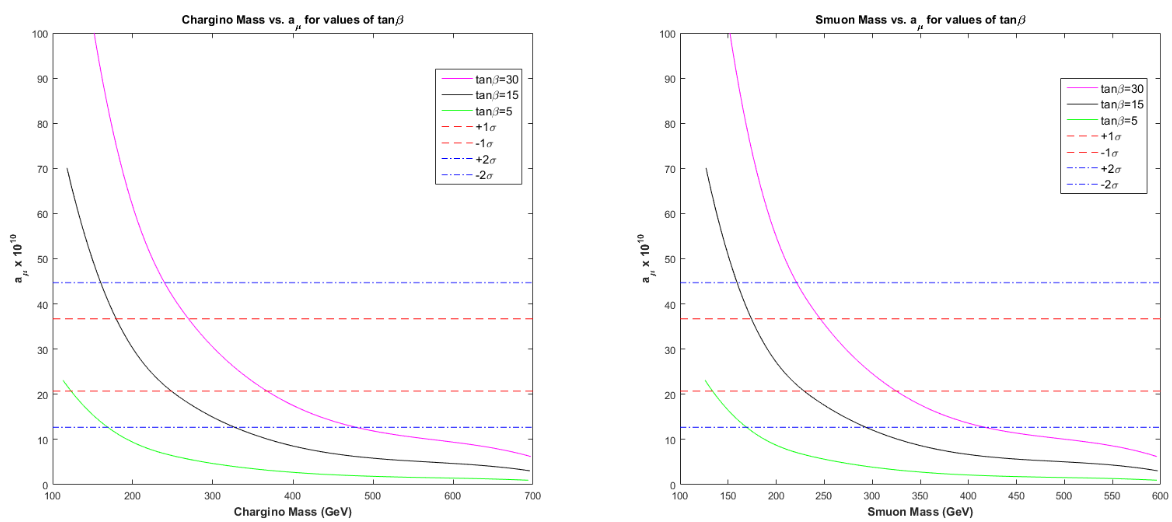


Figure 8: a_μ arising from supersymmetry as a function of lightest chargino and smuon masses, for different values of $\tan\beta$. Horizontal lines show the 1 and 2σ bounds from E821.

shown. From this we can easily identify values of the various parameters that are consistent with the experimental results. Of course, there are many other experimental constraints that would also need to be satisfied.

As another example, consider a simplified scenario where we assume that all the superpartners have the same mass, M_{SUSY}. In that case, eqs. (61) and (62) can be combined and simplified to:

$$\begin{aligned}\delta a_\mu^{SUSY} &= \frac{\tan\beta}{192\pi^2}\frac{m_\mu^2}{M_{SUSY}^2}\left(5g_2^2+g_1^2\right)\\ &= 14\tan\beta\left(\frac{100\ GeV}{M_{SUSY}}\right)^2 10^{-10}\end{aligned}$$

Now the results depend only on two parameters, $\tan\beta$ and M_{SUSY}, instead of all the parameters we defined earlier. Fig. 9 shows the ranges of $\tan\beta$ and M_{SUSY} that correspond to the observed

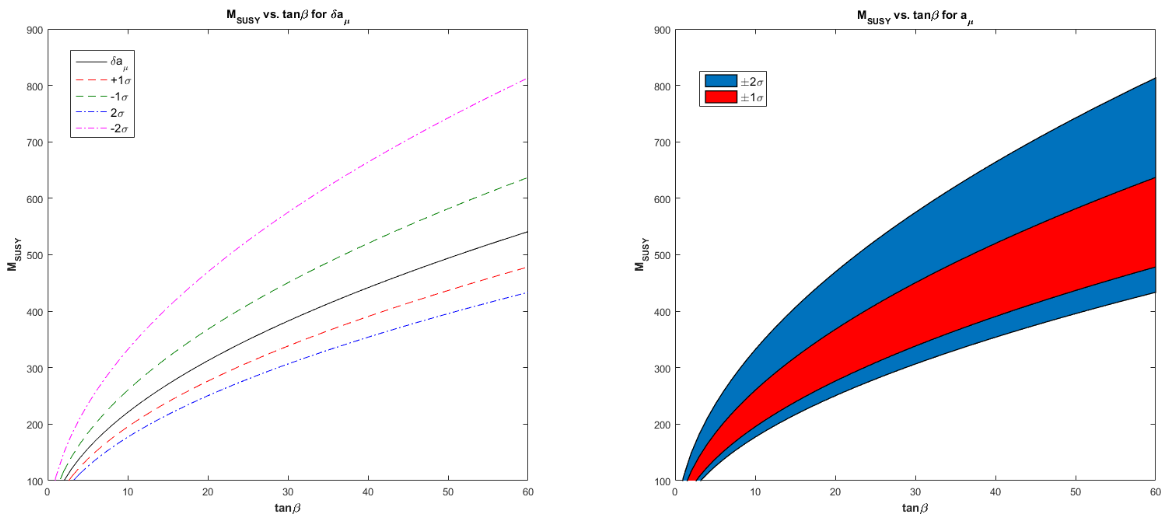


Figure 9: SUSY contribution to a_μ assuming all superpartners have the same mass, as a function of $\tan\beta$ and M_{SUSY}. The solid line on the left shows the E821 central value; dashed lines show the $1s$ and 2σ bounds.

discrepancy in a_μ (solid line) and the 1σ and $2sigma$ ranges. One expects that $3 < \tan\beta < 60$ and M_{SUSY} must be at least $100\ GeV$ to evade the direct detection bounds coming from the LHC.

Therefore, if we believe that the discrepancy between the values is at most 1σ from what was actually measured, we can constrain the mass of the superpartners and use that in our model.

From this, we could proceed to search for different models that match the a_μ discrepancy and perhaps discover the true underlying theory beyond the Standard Model. However, these models must also satisfy other experimental constraints, like the Higgs mass, coming from searches at the LHC and any earlier experiments. Currently, major experiments at CERN, Fermilab, and around the world are being run to try and uncover the new principles that take us beyond the Standard Model.

Furthermore, at Fermilab, the E989 experiment is set to begin collecting data this year on new measurements for a_μ. The experiment aims to be much more precise than the E821 experiment and therefore lower the uncertainty on δa_μ. Therefore, in a couple of years, the more precise measurements may close the gap between theory and experiment, or they may widen that gap and provide considerable indirect evidence that there are aspects of our universe still yet to be discovered and understood.

Appendix A Commutator of the Covariant Derivative

Here, we show that $i[D_\mu, D_\nu] = eF_{\mu\nu}$. Since $D_\mu = \partial_\mu - ieA_\mu$, we have:

$$i[D_\mu, D_\nu] = i(\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu)$$

It should be imagined that this acts on a function standing to the right of our expression. To simplify this, I will use the fact that:

$$[\partial_\mu, \partial_\nu] = 0 \quad [A_\mu, A_\nu] = 0$$

leaving:

$$i[D_\mu, D_\nu] = i(-ie)(\partial_\mu A_\nu - A_\nu\partial_\mu + A_\mu\partial_\nu - \partial_\nu A_\mu)$$

Next, we recall that this expression is actually multipled by a function, ψ. Therefore, we can distribute this function and then use the product rule to obtain:

$$i[D_\mu, D_\nu]\psi = e(\psi\partial_\mu A_\nu + A_\nu\partial_\mu\psi + A_\mu\partial_\nu\psi - A_\nu\partial_\mu\psi - \psi\partial_\nu A_\mu - A_\mu\partial_\nu\psi)$$

We cancel terms and conclude:

$$i[D_\mu, D_\nu]\psi = e(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi$$

However, $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$ which leads to the final result:

$$i[D_\mu, D_\nu] = eF_{\mu\nu}$$

Appendix B The Gordon Decomposition

To prove the Gordon Decomposition, we first start with the expression:

$$\overline{u}(p')(\not{p}'\gamma^\mu + \gamma^\mu \not{p})u(p) \tag{63}$$

We can manipulate this expression in two different ways that will ultimately lead to the final result of the Gordon decomposition used in our QED one loop correction.

The first way will use our definitions of $g^{\mu\nu}$ and $\sigma^{\mu\nu}$ to put eq. (63) in terms of these values. We first rewrite eq. (63):

$$\overline{u}(p')(p'_\nu \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu p_\nu)u(p)$$

It can be shown using our definitions of the gamma matrices and $\sigma^{\mu\nu}$ that:

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$$

Using this relation results in:

$$\overline{u}(p')[p'_\nu(g^{\nu\mu} - i\sigma^{\nu\mu}) + (g^{\mu\nu} - i\sigma^{\mu\nu})p_\nu]u(p)$$

We know that $\sigma^{\mu\nu}$ is defined to be antisymmetric so using that fact and rearranging terms, we obtain eq. (63) is:

$$\overline{u}(p')[(p' + p)^\mu + i\sigma^{\mu\nu}(p' - p)_\nu]u(p) \tag{64}$$

Our second way of manipulating eq. (63) is to use the Dirac equation which says:

$$(\not{p} - m)u(p) = 0 \qquad \overline{u}(p)(\not{p} - m) = 0$$

This allows to rewrite eq. (63) as:

$$\overline{u}(p')[m\gamma^\mu + \gamma^\mu m]u(p)$$

which leaves us with:

$$2m\overline{u}(p')\gamma^\mu u(p) \tag{65}$$

Finally, since eq. (64) and eq. (65) are both equivalent to eq. (63), we can set them equal to each other and get the equation for the Gordon Decomposition:

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left[\frac{(p' + p)^\mu}{2m} + i\sigma^{\mu\nu}\frac{(p' - p)_\nu}{2m}\right]u(p) \tag{66}$$

Appendix C Feynman Rules

Here, we state the Feynman rules for QED and the electroweak theory, used in our calculations.

C.1 Feynman Rules for QED

First, on the associated diagram, each line must be notated with a specific momentum in a specific direction and each vertex must be given a contravariant index, like $\mu, \nu, etc.$ We then start along the fermion lines and work our way backwards, having outgoing electron/muons contribute a $\overline{u}$ and incoming contribute a u. Along the fermion lines, each vertex between two fermions and a photon contributes a factor:

$$ie\gamma^{\mu}$$

and each internal electron/muon line contributes:

$$\frac{i}{\not{k} - m}$$

where k is the momentum of that internal line. After moving along the fermion lines, we add on any factors from internal photons:

$$\frac{-ig_{\mu\nu}}{k^2}$$

where again k is the momentum of the photon and the indexes associated with the Minkowski metric result from the two vertices of the internal photon. We then for each vertex ensure conservation of energy and momentum by adding a delta function term:

$$(2\pi)^4\delta^4(p + k - q)$$

If the particle is moving towards the vertex, its momentum will be positive and if it is moving away, we make it negative. Furthermore, for every internal line, we integrate over its four momentum, contributing:

$$\frac{d^4k}{(2\pi)^4}$$

After integrating over these internal lines' momenta, we will be left with one integral for each closed loop in our diagram. We will also be left with one delta function multiplied by $(2\pi)^4$ which we proceed to then drop. Finally, we multiply the entire expression by a $-i$.

When using the Feynman rules, it is of the utmost importance that one follows the fermion lines backward first so that one gets the correct mathematical expression (row·matrix·column) under the integral. Following these rules, one can arrive at eq.(14) and then use it to find the anomalous magnetic moment of a fermion, as done in this thesis.

C.2 Feynman Rules for EW Theory

For EW theory, we work in a general renormalizable gauge where $\xi = 1$. [4]. The rules here will follow the same procedure as with QED but with different factors from vertices and internal lines. The first interaction we investigate is that involving the Z boson. The vertex factor for a lepton coupled to the Z boson is:

$$\frac{-ig}{4\cos\theta_w}\gamma^\mu\left[(1-4\sin^2\theta_w)-\gamma^5\right]$$

Here, we have introduced g and θ_w, the weak coupling constant and the weak mixing angle respectively. The weak mixing angle is a constant that relates the masses of the Z and W boson, which we will numerically evaluate later. Furthermore, we have introduced a new gamma matrix, γ^5, defined as:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

This unique gamma matrix will appear throughout our calculation of the weak corrections. Next, the propagator for the Z boson is:

$$\frac{-ig_{\mu\nu}}{k^2-M_Z^2}$$

where k is the momentum and M_Z is the mass of the Z boson.

The Z Goldstone boson has its own unique rules, as well. Its propagator term will be:

$$\frac{i}{k^2-M_Z^2}$$

and its vertex factor becomes:

$$\frac{gm}{2M_W}\gamma^5$$

where m is the mass of the lepton and M_W is the W boson's mass.

We also calculate the contribution from the W boson. Since it is part of the EW theory, it will also have Goldstone boson diagrams. The W boson is unique in that each vertex with a lepton will also be connected to a neutrino. For the W boson, the vertex factor if it is connected to a lepton and neutrino is:

$$\frac{-ig}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

where g is again the weak coupling constant. The propagator is:

$$\frac{-ig_{\mu\nu}}{k^2-M_W^2}$$

the same as the Z boson's term, but with M_W, the mass of the W boson, switched in for M_Z. If the W boson is coupled to another W boson and a photon, however, the vertex factor is:

$$-ie\left[g_{\mu\nu}(q'+k')_\alpha - g_{\nu\alpha}(k'+q)_\mu + g_{\alpha\mu}(q-q')_\nu\right]$$

where q' is the momentum of the incoming W boson, q is the momentum of the incoming photon, and k' is the momentum of the outgoing W boson.

For the W Goldstone bosons, the propagator term is:

$$\frac{i}{k^2 - M_W^2}$$

which is the same as for the Z Goldstone boson with M_W substituted in for M_Z. There are four new vertex factors to define. The first two are for a vertex connecting a lepton, neutrino, and Goldstone boson:

$$\frac{-ig}{2\sqrt{2}}\left(1 \pm \gamma^5\right)\frac{m}{M_W}$$

The sign associated with the γ^5 will be negative if the neutrino is incoming and the lepton is outgoing, positive if they are flipped. The third vertex factor results from connecting a Goldstone boson, a photon, and a W boson:

$$-ieM_W g_{\mu\nu}$$

Lastly, a connection between two Goldstone bosons and a photon has a vertex factor of:

$$-ie(p+p')^\mu$$

where p is the momentum of the incoming Goldstone boson and p' is the momentum of the outgoing Goldstone boson. We can now use these new rules to calculate the contributions to a_μ from the Z boson and W boson, starting with the Z boson.

Appendix D Contraction Theorems

In many of our calculations to extract a_μ, we have made use of various contraction theorems that help us to eliminate gamma matrices. I will now prove to you two of them specifically and will leave it to the reader to prove the rest that I list.

From the anticommutation relation, we have a list of contraction theorems:

$$\begin{aligned}
&(1) \quad \gamma_\mu \gamma^\mu = 4 \\
&(2) \quad \gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu \\
&(3) \quad \gamma_\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\nu\lambda} \\
&(4) \quad \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu
\end{aligned}$$

The first that I will prove is (2):

$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$$

We will use the anticommutation relation for gamma matrices to prove this:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

We multiply both sides by γ^μ and simplify:

$$\gamma_\mu \gamma^\mu \gamma^\nu + \gamma_\mu \gamma^\nu \gamma^\mu = 2\gamma_\mu g^{\mu\nu}$$

From (1) and acting with the Minkowski metric:

$$4\gamma^\nu + \gamma_\mu \gamma^\nu \gamma^\mu = 2\gamma^\nu$$

Getting the γ^ν's on the same side, we conclude:

$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$$

The other one that I wll prove is (4). As before, we will make use of the anticommutation relation:

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\lambda (2g^{\sigma\mu} - \gamma^\mu \gamma^\sigma)$$

Multiplying out our terms:

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = 2\gamma^\sigma \gamma^\nu \gamma^\lambda - \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\sigma$$

Using (3) and the anticommutation relation again:

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = 2\gamma^\sigma(2g^{\nu\lambda} - \gamma^\lambda \gamma^\nu) - 4g^{\nu\lambda}\gamma^\sigma)$$

Multiplying out terms and canceling the γ^σ terms, we obtain:

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$$

References

[1] P. A. M. Dirac, “The Quantum theory of the electron,” Proc. Roy. Soc. Lond. A **117**, 610 (1928).

[2] J. S. Schwinger, “Quantum electrodynamics. III: The electromagnetic properties of the electron: Radiative corrections to scattering,” Phys. Rev. **76**, 790 (1949).

[3] A. Zee, “Quantum field theory in a nutshell,” Princeton, UK: Princeton Univ. Pr. (2010) 576p

[4] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory,” Reading, USA: Addison-Wesley (1995) 842p

[5] A. Hoecker and W. J. Marciano in: K. A. Olive *et al.* [Particle Data Group Collaboration], “Review of Particle Physics,” Chin. Phys. C **38**, 090001 (2014).

[6] T. Blum, A. Denig, I. Logashenko, E. de Rafael, B. Lee Roberts, T. Teubner and G. Venanzoni, “The Muon $(g-2)$ Theory Value: Present and Future,” [arXiv:1311.2198].

[7] S. P. Martin and J. D. Wells, “Muon anomalous magnetic dipole moment in supersymmetric theories,” Phys. Rev. D **64**, 035003 (2001)

[8] B. C. Allanach, “SOFTSUSY: a program for calculating supersymmetric spectra,” Comput. Phys. Commun. **143**, 305 (2002)

[9] B. C. Allanach, S. P. Martin, D. G. Robertson and R. R. de Austri, “The Inclusion of Two-Loop SUSYQCD Corrections to Gluino and Squark Pole Masses in the Minimal and Next-to-Minimal Supersymmetric Standard Model: SOFTSUSY3.7,” arXiv:1601.06657 [hep-ph].