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Gary McCartor  
*Southern Methodist University*

David G. Robertson  
*Otterbein University*

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# The Mandelstam-Leibbrandt Prescription in Light-Cone Quantized Gauge Theories

Gary McCartor

*Department of Physics, Southern Methodist University, Dallas, TX 75275*

David G. Robertson

*Department of Physics, The Ohio State University, Columbus, OH 43210*

## Abstract

Quantization of gauge theories on characteristic surfaces and in the light-cone gauge is discussed. Implementation of the Mandelstam-Leibbrandt prescription for the spurious singularity is shown to require two distinct null planes, with independent degrees of freedom initialized on each. The relation of this theory to the usual light-cone formulation of gauge field theory, using a single null plane, is described. A connection is established between this formalism and a recently given operator solution to the Schwinger model in the light-cone gauge.

## I. INTRODUCTION

Like the other axial gauges, the light-cone gauge ( $A^+ \equiv A^0 + A^3 = 0$ )<sup>1</sup> is plagued by the occurrence of a “spurious” singularity in the gauge field propagator, which is related to the residual gauge freedom (in this case transformations that do not depend on  $x^-$ ). For the light-cone gauge a consistent interpretation of this singularity seems to be via the Mandelstam-Leibbrandt (ML) prescription [1,2]. A large number of calculations using this prescription have been performed and all give sensible results, in agreement, where comparison is possible, with covariant-gauge calculations. For a good overview of these and related issues see Ref. [3].

A derivation of the ML form of the propagator has been given in Ref. [4], in the framework of equal-time canonical quantization. It has further been shown that gauge theories formulated in this way are renormalizable [5] (although some nonlocal counterterms are required). A central feature of this formalism is that one does not reduce completely down to the physical (transverse) degrees of freedom. A longitudinal component of the gauge field is retained, and a corresponding ghost field. The Hilbert space of the theory thus possesses an indefinite metric. Selection of a physical subspace results in the recovery of a positive-semidefinite metric and Poincaré invariance.

Light-cone quantization of this theory was discussed in Ref. [6], where it was shown that a second characteristic surface, nowhere parallel to the conventional light-cone initial-value surface  $x^+ = 0$ , is needed to correctly recover the ML form of the propagator. The unphysical fields in the theory are initialized along this other surface, and proper attention must be paid to the inclusion of boundary contributions in the construction of conserved charges, for example, the Poincaré generators. Thus we do not have a strictly “Hamiltonian” formalism, with all fields evolving from a single initial-value surface. This type of situation is quite familiar from the treatment of massless fields quantized on characteristic surfaces, particularly in two spacetime dimensions [7].

The discussion of Ref. [6] was for simplicity limited to free fields. This was sufficient for identifying the relevant degrees of freedom and determining the propagator, which was the object of primary interest. The purpose of the present paper is to show how the construction of Ref. [6] is generalized to interacting theories in a simple Abelian context. We shall begin by reviewing the free gauge field quantized on characteristic surfaces. We then discuss the simplest extension of this, quantum electrodynamics. Most of the features present in more complicated cases (e.g., non-Abelian gauge theory) are already present in QED, and it further allows us to discuss the treatment of Fermi fields in the simplest nontrivial setting. The case of two spacetime dimensions is instructive in that it requires some special treatment, and furthermore is exactly solvable for vanishing fermion mass. In Sect. 3 we establish a connection between the formalism presented here and an operator solution to the Schwinger model in the light-cone gauge given recently by Nakawaki [8].

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<sup>1</sup>We define light-cone coordinates  $x^\pm \equiv x^0 \pm x^3$  and use latin indices  $(i, j, \dots)$  to index the transverse components  $x_\perp = (x^1, x^2)$ . A contraction of four-vectors decomposes as  $A \cdot B = \frac{1}{2}(A^+B^- + A^-B^+) - A^iB^i$ , from which we infer the metric  $g_{+-} = g_{-+} = \frac{1}{2}$ ,  $g^{11} = g^{22} = -1$ , with all other components vanishing. Derivatives are defined by  $\partial_\pm \equiv \partial/\partial x^\pm$ ,  $\partial_i \equiv \partial/\partial x^i$ .

As we shall see, the central problem is that of determining the algebra of the field operators. It will prove to be quite difficult to find a set of commutation relations that result in the Heisenberg equations exactly reproducing the equations of motion. Thus we shall be unable to construct an interacting theory quantized on characteristics that is precisely isomorphic to the theory described in Ref. [4]. It is possible, however, to construct a theory that is equivalent to the full theory on the physical subspace. A field redefinition, which has essentially the form of a residual gauge transformation, allows us to use simple (free-field) commutation relations to achieve this. The resulting theory is simply the “naive” light-cone theory tensored with the unphysical fields, which are now decoupled. They may therefore be discarded by invoking the physical subspace condition. In this way we obtain a better understanding of the relation between the ordinary light-cone formulation of gauge theories and the formulation with the ML prescription implemented.

## II. QUANTUM ELECTRODYNAMICS

We begin by recalling the main results of Ref. [6]. The light-cone gauge is parameterized by  $n_\mu A^\mu = 0$  with  $n_+ = 1$  and  $n_- = n_\perp = 0$ . For a free gauge field we consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda n_\mu A^\mu, \quad (2.1)$$

where  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ , and  $\lambda$  is a Lagrange multiplier field whose equation of motion enforces the gauge condition  $A^+ = 0$ . It satisfies

$$\partial_- \lambda = 0, \quad (2.2)$$

which indicates that it must be initialized along a surface of constant  $x^-$ , rather than the usual surface  $x^+ = 0$ . It turns out to be conjugate to a field  $\varphi$ , which is related to the value of the transverse field  $A^i$  at longitudinal infinity ( $x^- = \pm\infty$ ). Thus  $\lambda$  and  $\varphi$  satisfy an equal- $x^-$  commutation relation, while the remainder of the transverse field has the conventional equal- $x^+$  commutator. Furthermore, the boundary contributions to the Poincaré generators (more generally, to all conserved charges) must be retained to correctly incorporate contributions from  $\lambda$  and  $\varphi$ . The Hilbert space of this theory has an indefinite metric and we must in the end project onto a physical subspace, in a way familiar from quantization in covariant gauges. Specifically, we define physical states to be those annihilated by the positive frequency part of  $\lambda$ :

$$\lambda^{(+)}|\text{phys}\rangle = 0, \quad (2.3)$$

that is, states between which  $\lambda$  has vanishing matrix elements. Maxwell’s equations and the Poincaré algebra are obtained in matrix elements between these states, which furthermore have nonnegative norm.

Let us now consider QED, defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda n_\mu A^\mu + \bar{\psi}(i\not{\partial} - m)\psi - gJ_\mu A^\mu \quad (2.4)$$

with  $J^\mu \equiv \bar{\psi}\gamma^\mu\psi$ . Our approach will be the same as that in Ref. [6]: use the equations of motion to identify the degrees of freedom and where they must be initialized, and then attempt to determine the field algebra by demanding that the Heisenberg equations correctly reproduce the Euler-Lagrange equations. Finally, we must check that projection onto a suitable physical subspace can be carried out in a consistent way.

The equations of motion that follow from the Lagrangian (2.4) are, with  $A^+ = 0$ ,

$$\partial_-^2 A^- + \partial_- \partial_i A^i = -\frac{1}{2}gJ^+, \quad (2.5)$$

$$2\partial_+ \partial_- A^- - \partial_\perp^2 A^- - 2\partial_+ \partial_i A^i = 2\lambda + gJ^-, \quad (2.6)$$

$$(4\partial_+ \partial_- - \partial_\perp^2)A^i + \partial_i(\partial_- A^- + \partial_j A^j) = gJ^i, \quad (2.7)$$

$$i\partial_- \psi_- = \frac{1}{2}\left(-i\alpha^i \partial_i + m\beta - g\alpha^i A^i\right)\psi_+, \quad (2.8)$$

$$i\partial_+ \psi_+ = \frac{1}{2}gA^- \psi_+ + \frac{1}{2}\left(-i\alpha^i \partial_i + m\beta - g\alpha^i A^i\right)\psi_-, \quad (2.9)$$

where we have defined the standard light-cone spinor projections  $\psi_\pm \equiv \frac{1}{2}\gamma^0\gamma^\pm\psi$ , and  $\alpha^i = \gamma^0\gamma^i$  and  $\beta = \gamma^0$  are the original Dirac matrices. In addition the field  $\lambda$  satisfies Eq. (2.2) even in the presence of interactions. The easiest way to see this is to apply  $\partial_\nu$  to both sides of the equation of motion

$$\partial_\mu F^{\mu\nu} = n^\nu \lambda + gJ^\nu \quad (2.10)$$

giving

$$2\partial_- \lambda + g\partial_\mu J^\mu = 0. \quad (2.11)$$

We then note that  $\partial_\mu J^\mu$  vanishes by the Dirac equation.<sup>2</sup>

Now because  $\lambda$  satisfies Eq. (2.2), it must be initialized along a surface of constant  $x^-$ , as in the free theory [6]. Its ‘‘conjugate momentum’’ can be identified by considering the light-cone Gauss’ law, Eq. (2.5). It is convenient to formally integrate this equation and express  $A^-$  in terms of ‘‘zero mode’’ degrees of freedom, that is, the integration constants that arise. We write

$$\partial_- A^- + \partial_i A^i = -g\frac{1}{2\partial_-}J^+ + \partial_\perp^2 \phi(x^+, x_\perp), \quad (2.12)$$

where  $1/\partial_-$  is some particular antiderivative and  $\phi$  is an arbitrary function of  $x^+$  and  $x_\perp$ . As emphasized in Ref. [6],  $\phi$  is part of the classical data required to determine the general

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<sup>2</sup>This assumes that we have defined the current operator in such a way that it is not anomalous, as is normally required for consistency.

solution of the field equations, and so corresponds to a degree of freedom in the quantum field theory. Because it satisfies  $\partial_- \phi = 0$  by definition, it must also be initialized on a surface of constant  $x^-$ . It turns out to be essentially conjugate to  $\lambda$ , as in the free theory; this will be shown in detail below.

Integrating Eq. (2.12) again results in

$$A^- = -\frac{1}{\partial_-} \partial_i A^i - g \frac{1}{2\partial_-^2} J^+ + (\partial_\perp^2 \phi) x^- + \gamma(x^+, x_\perp), \quad (2.13)$$

where  $\gamma$  is another apparently arbitrary integration constant. As in the free theory, however, there will be a constraint relating the three zero mode fields  $\lambda$ ,  $\phi$ , and  $\gamma$ , so that only two of them are independent. We shall here take  $\lambda$  and  $\phi$  to be the independent quantities, and  $\gamma$  to be the determined one.

In addition to  $\lambda$  and  $\phi$ , there are of course degrees of freedom associated with the transverse fields  $A^i$ ; we shall return to these below. For the moment let us discuss the degrees of freedom associated with the Fermi field. In the usual light-cone treatment we observe that if  $\psi_+$  is specified on  $x^+ = 0$ , then Eq. (2.8) is an equation of constraint that determines  $\psi_-$ . Thus the actual fermionic degrees of freedom are contained in  $\psi_+$ . This is known to be correct for the free massive Fermi field.<sup>3</sup> Regarding the question of whether or not to include an arbitrary  $x^-$ -independent function in the solution of Eq. (2.8) for  $\psi_-$ , we note that any solution of the free massive Dirac equation that is independent of  $x^-$  has infinite energy, and so is presumably unphysical. Solutions of this type have been discussed recently [9], but they do not seem to be necessary in the construction of the free theory. We shall here assume that this holds true for interacting fields as well, and treat the Fermi field in the conventional way. This includes taking the field to vanish at longitudinal infinity when constructing conserved charges. Again, this is known to be the correct procedure in the free theory.

This boundary condition on  $\psi$  insures that the current  $J^+ = 2\psi_+^\dagger \psi_+$  vanishes at longitudinal infinity. It will be necessary, however, to impose the further condition that  $J^+$  have no zero mode, i.e., that

$$\int_{-\infty}^{\infty} dx^- J^+ = 0. \quad (2.14)$$

This is necessary, for example, to insure that the Hamiltonian density is integrable over  $x^+ = 0$ , as we shall see below. This condition on  $J^+$  also has implications for possible definitions of  $\partial_-^{-(1,2)} J^+$  in, e.g., Eqs. (2.12) and (2.13). We might define, for example,

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<sup>3</sup>In the massless case one can worry about modes with  $k^+ = k_\perp = 0$ , which represent quanta propagating precisely along the surface  $x^+ = 0$ . They therefore cannot be initialized there, and additional information must be given on another characteristic surface to make the theory complete. This is certainly important in two spacetime dimensions, where the modes under discussion constitute half of the theory. In 3+1 dimensions these modes are a set of measure zero and are conventionally neglected. Some further discussion of this problem will be given when we consider the transverse gauge field.

$$\frac{1}{\partial_-} J^+(x^-) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dy^- \epsilon(x^- - y^-) J^+(y^-) \quad (2.15)$$

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \quad (2.16)$$

which is the Cauchy principal value in momentum space. Iteration of this to obtain  $\partial_-^{-2}$  is in general ill-defined, however, unless the integrand has no zero mode. The condition (2.14) insures that essentially any antiderivative may be used consistently on  $J^+$ .

We may implement this condition on the current simply by coupling  $A^-$  to

$$J^{+'} \equiv J^+ - \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L dx^- J^+, \quad (2.17)$$

instead of to  $J^+$  directly. We do not yet understand all of the implications of this redefinition on the structure of the theory. It resembles the choice of a restricted space of test functions for the elementary fields in light-cone quantization [see the discussion following Eq. (2.26)], but applied to a composite object. Since it involves the definition of the current operator, its effects may be expected to depend on the particular regulator employed and the methods used to define operator products. It is thus difficult to make general statements.<sup>4</sup>

Finally let us discuss the transverse fields  $A^i$ . We begin by observing that the field  $\phi$  is related to the value of  $A^i$  on the longitudinal boundaries, which we denote by  $A_\infty^i$ :

$$A_\infty^i \equiv \lim_{x^- \rightarrow \pm\infty} A^i. \quad (2.18)$$

The most physically motivated way of seeing this is to consider the classical expressions for the energy and momentum and demand that these be finite. We have

$$P^\mu = \frac{1}{2} \int_a dx^- d^2 x_\perp T^{+\mu} + \frac{1}{2} \int_b dx^+ d^2 x_\perp T^{-\mu}, \quad (2.19)$$

where  $a$  is the usual light-cone initial value surface  $x^+ = 0$  and  $b$  are the boundary wings (see Fig. 1). The need to retain the contributions from the boundary surfaces is in fact quite general, and follows from insisting that the generators be the same as those we would construct in equal-time quantization [7]. For the energy-momentum tensor we may take the gauge-invariant form

$$T^{\mu\nu} = -F^{\mu\lambda} F_\lambda^\nu - g J^\mu A^\nu + i \bar{\psi} \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \mathcal{L} - \lambda n^\mu A^\nu. \quad (2.20)$$

Now for the four-momentum (2.19) to be finite, a necessary condition is that those components of the field strength  $F_{\mu\nu}$  that appear in the integral over the surface  $x^+ = 0$  must vanish

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<sup>4</sup>The necessity of the limiting procedure in Eq. (2.17), and indeed this entire discussion, suggests that it may be natural to consider the theory defined on a finite interval in  $x^-$ , with suitable boundary conditions imposed on the fields. This regulates the infrared behavior and allows a rigorous discussion of these issues, up to the treatment of UV divergences. This type of approach is discussed in Refs. [10,11].

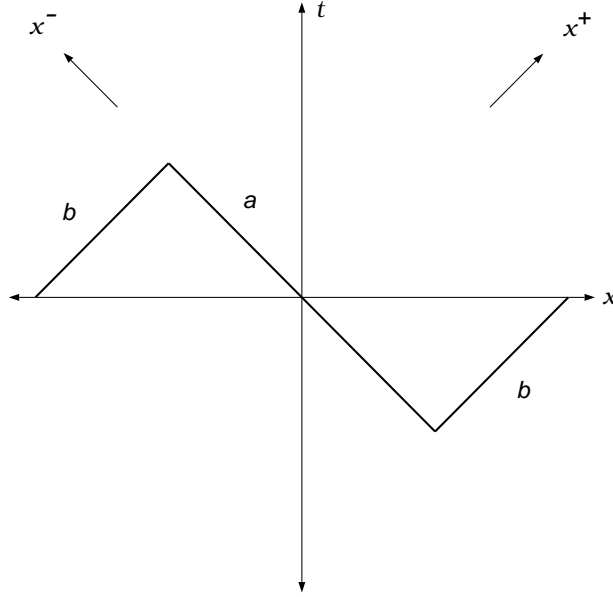


FIG. 1. Standard light-cone initial-value surface  $x^+ = 0$  (a), and boundary wings (b).

at longitudinal infinity. These are  $F^{+-}$ ,  $F^{+i}$ , and  $F^{ij}$ , as may be seen from an inspection of Eq. (2.20). We consider first the component

$$\begin{aligned} \frac{1}{2}F^{+-} &= \partial_- A^- \\ &= -\partial_i A^i - g \frac{1}{2\partial_-} J^+ + \partial_\perp^2 \phi(x^+, x_\perp), \end{aligned} \quad (2.21)$$

where we have made use of Eq. (2.12). Requiring that this vanish on the boundary wings results in

$$\partial_i A_\infty^i = \partial_\perp^2 \phi \quad (2.22)$$

(recall that the current and its antiderivative are taken to vanish on the boundaries). Thus

$$A_\infty^i = \partial_i \phi. \quad (2.23)$$

Not surprisingly,  $A_\infty^i$  has the form of a pure gauge. It follows that  $F^{+i}$  and  $F^{ij}$  will also vanish on the boundary surfaces.

The fact that  $A^i$  goes to a pure gauge at longitudinal infinity suggests a natural field redefinition to disentangle the fields that are initialized on  $x^+ = 0$  from those that live on the boundaries. Its form is simply that of a gauge transformation that removes the boundary value of  $A^i$ . Specifically we define



$$A^i \equiv T^i + \partial_i \phi . \quad (2.24)$$

It is consistent to take  $T^i$  to vanish on the boundaries, as is clear from the arguments presented above. This can also be seen from the equation of motion for the transverse fields. Inserting Eq. (2.12) into Eq. (2.7) we obtain

$$(4\partial_+ \partial_- - \partial_\perp^2) A^i + \partial_i \partial_\perp^2 \phi - g \frac{1}{2\partial_-} \partial_i J^+ = g J^i , \quad (2.25)$$

from which we see again that it is not consistent to assume that  $A^i$  vanishes on the boundaries: the presence of the term  $\partial_i \partial_\perp^2 \phi$  means that even if  $A^i$  is initially zero at  $x^- = \pm\infty$  it will not remain so under evolution in  $x^+$ . In terms of the redefined field, however, we have

$$(4\partial_+ \partial_- - \partial_\perp^2) T^i - g \frac{1}{2\partial_-} \partial_i J^+ = g J^i . \quad (2.26)$$

It is thus consistent to assume that  $T^i$  vanishes at longitudinal infinity for all  $x^+$ . As with massless fields generally, one can consider the modes of  $T^i$  that have  $k^+ = k_\perp = 0$ . (Note that these are a set of measure zero even relative to the fields  $\lambda$  and  $\phi$ .) This is a familiar problem in the light-cone quantization of massless fields, which is conventionally treated by choosing a test function space in which to smear the field operators that has vanishing support at the point  $k^+ = 0$  [12]. Thus they can be consistently neglected, with the additional consequence that the integral operator  $1/\partial_-$  will be well-defined when acting on  $T^i$ , since it has no  $k^+ = 0$  modes.

Given  $T^i$  and  $\psi_+$  on  $x^+ = 0$ , and  $\phi$ ,  $\lambda$ , and  $\gamma$  on the boundary surfaces, we appear to have sufficient data to determine the solution to Eqs. (2.9) and (2.26) and solve for the independent fields everywhere in spacetime. All that remains is to insure that this solution is consistent with Ampere's law, Eq. (2.6). Substituting in the field redefinition (2.24), and making use of Eq. (2.26), we find that Eq. (2.6) reduces to a constraint relating the three zero mode fields. The precise form that this constraint takes depends on the definition of the current, and in particular on the way the gauge field becomes mixed with the Fermi field under renormalization. This will in general depend on the regulator used. Here we shall assume that the zero mode fields do not mix with the current, as would presumably be true for calculations using a gauge-invariant regulator such as dimensional regularization. In Sect. 3 we shall show explicitly for the case of two dimensions that such mixing can happen and that it modifies the details of the constraint for the zero mode fields. With the assumption of no mixing of the zero modes, however, we obtain

$$\frac{1}{2} \gamma + \partial_+ \phi + \frac{1}{\partial_\perp^2} \lambda = 0 . \quad (2.27)$$

As noted previously, we shall take  $\phi$  and  $\lambda$  to be the independent quantities and  $\gamma$  to be determined by Eq. (2.27).

The final result of our analysis of the equations of motion is that specifying  $T^i$  and  $\psi_+$  on  $x^+ = 0$ , and  $\phi$  and  $\lambda$  on the boundary wings, is sufficient to determine a general solution to the field equations (2.5)–(2.9). These fields will thus correspond to independent operators in the quantum theory. The algebra they satisfy may be determined by considering the

Poincaré generators, and requiring that the Heisenberg equations correctly reproduce the field equations.

We focus on  $P^\pm$ , for which the relevant components of the energy-momentum tensor (2.20) are

$$T^{++} = 4(\partial_- A^i)^2 + 4i\psi_+^\dagger \partial_- \psi_+ , \quad (2.28)$$

$$T^{+-} = (\partial_- A^-)^2 + \frac{1}{2}(F^{ij})^2 + \psi_-^\dagger (-i\alpha^i \partial_i + m\beta - g\alpha^i A^i) \psi_+ + \text{H.c.} , \quad (2.29)$$

$$T^{--} = 4(\partial_+ A^i)^2 + 4(\partial_+ A^i)(\partial_i A^-) + (\partial_i A^-)^2 - 2\lambda A^- + 4i\psi_-^\dagger \partial_+ \psi_- , \quad (2.30)$$

$$\begin{aligned} T^{-+} = & (\partial_- A^-)^2 + \frac{1}{2}(F^{ij})^2 - 4i\psi_+^\dagger \partial_+ \psi_+ - 2i\psi_+^\dagger \alpha^i \partial_i \psi_- - 2i\psi_-^\dagger \alpha^i \partial_i \psi_+ \\ & + 2m(\psi_-^\dagger \beta \psi_+ + \psi_+^\dagger \beta \psi_-) + gJ^+ A^- - 2gJ^i A^i . \end{aligned} \quad (2.31)$$

We begin by expressing these, and the equations of motion (2.9) and (2.26), in terms of the independent fields  $T^i$ ,  $\psi_+$ ,  $\phi$ , and  $\lambda$ , keeping in mind that  $T^i$  and  $\psi_\pm$  may be set to zero on the boundary surfaces. The resulting expressions are simplified considerably, however, by making the further field redefinition

$$\psi_+ \equiv e^{ig\phi} \eta_+ . \quad (2.32)$$

For  $A^i$  and  $\psi_+$  the redefinitions (2.24) and (2.32) have the form of a residual ( $x^-$ -independent) gauge transformation with the gauge function  $\phi$ .<sup>5</sup> Note that the projection of the Dirac equation that determines  $\psi_-$  becomes, in terms of  $\eta_+$  and  $T^i$ ,

$$i\partial_- \psi_- = e^{ig\phi} \frac{1}{2} \left( -i\alpha^i \partial_i + m\beta - g\alpha^i T^i \right) \eta_+ . \quad (2.34)$$

Thus the natural definition

$$\psi_- \equiv e^{ig\phi} \eta_- \quad (2.35)$$

results in  $\eta_-$  satisfying the conventional constraint relation, but written in terms of  $\eta_+$  and  $T^i$ . Furthermore, the currents have the same form in terms of  $\eta_\pm$  as they do in terms of  $\psi_\pm$ .

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<sup>5</sup>It is interesting in this connection to examine the solution of Gauss' law for  $A^-$  in terms of the independent fields:

$$\begin{aligned} A^- = & -\frac{1}{\partial_-} \partial_i T^i - g \frac{1}{2\partial_-^2} J^+ - 2\partial_+ \phi - \frac{2}{\partial_\perp^2} \lambda \\ \equiv & T^- - 2\partial_+ \phi - \frac{2}{\partial_\perp^2} \lambda . \end{aligned} \quad (2.33)$$

Except for the term involving  $\lambda$ , this can be interpreted as the corresponding gauge transformation of  $A^-$ , where the “transformed” field  $T^-$  is simply the naive expression for the constrained component of the gauge field, written in terms of  $T^i$  rather than  $A^i$ .

In terms of the redefined fields we find

$$\begin{aligned}
T^{+-} = & \eta_+^\dagger \left( -i\alpha^i \partial_i + m\beta - g\alpha^i T^i \right) \frac{1}{2i\partial_-} \left( -i\alpha^j \partial_j + m\beta - g\alpha^j T^j \right) \eta_+ + \text{H.c.} \\
& + (\partial_i T^j)(\partial_i T^j) + g(\partial_i T^i) \left( \frac{1}{\partial_-} J^+ \right) + \frac{g^2}{4} \left( \frac{1}{\partial_-} J^+ \right)^2, \tag{2.36}
\end{aligned}$$

where we have dropped terms that will not contribute when we integrate over  $x^+ = 0$  to construct  $P^-$ . The resulting contribution to  $P^-$  is just the naive light-cone Hamiltonian for QED, written in terms of  $T^i$  rather than  $A^i$ . As anticipated, all coupling between the physical fields  $T^i$  and  $\eta$  and the zero modes has disappeared from  $T^{+-}$ . Similarly

$$T^{++} = 4(\partial_- T^i)^2 + 4i\eta_+^\dagger \partial_- \eta_+. \tag{2.37}$$

The components  $T^{-+}$  and  $T^{--}$ , which are needed only on the boundary surfaces, are precisely the same as in the free theory. We find that

$$T^{--} = 4\lambda(\partial_+ \phi) \tag{2.38}$$

and

$$T^{-+} = 0 \tag{2.39}$$

on the boundaries.

The equations of motion are also easily rewritten in terms of the redefined fields. The solution to Gauss' law is

$$A^- = -\frac{1}{\partial_-} \partial_i T^i - g \frac{1}{2\partial_-^2} J^+ - 2\partial_+ \phi - 2 \left( \frac{1}{\partial_\perp^2} \lambda \right), \tag{2.40}$$

and the equation of motion for the transverse field  $T^i$  is Eq. (2.26). Finally, the dynamical part of the Dirac equation [Eq. (2.9)] becomes

$$\begin{aligned}
i\partial_+ \psi_+ = & \frac{1}{2} \left( -i\alpha^i \partial_i + m\beta - g\alpha^i T^i \right) \frac{1}{2i\partial_-} \left( -i\alpha^i \partial_i + m\beta - g\alpha^i T^i \right) \eta_+ \\
& - \frac{1}{2} g \left( \frac{1}{\partial_-} \partial_i T^i \right) \eta_+ - g \left( \frac{1}{\partial_\perp^2} \lambda \right) \eta_+ - \frac{1}{4} g^2 \left( \frac{1}{\partial_-^2} J^+ \right) \eta_+. \tag{2.41}
\end{aligned}$$

Notice that the only difference between Eq. (2.41) and what we would obtain in a naive light-cone quantization of QED is the term involving  $\lambda$ .

We can now attempt to determine the commutation relations among the fields such that Eqs. (2.26) and (2.41) are obtained in appropriate commutators of  $T^i$  or  $\eta_+$  with  $P^-$ . In addition, they must give the correct results for commutators with the ‘‘kinematical’’ generators, for example  $P^+$ , which translate the fields within their respective initial-value surfaces. The free-field commutation relations [6]

$$\left[ T^i(x^-, x_\perp), \partial^+ T^j(y^-, y_\perp) \right]_{x^+=0} = i\delta^{ij}\delta(x^- - y^-)\delta^{(2)}(x_\perp - y_\perp) \quad (2.42)$$

$$\left[ \phi(x^+, x_\perp), \lambda(y^+, y_\perp) \right] = i\delta(x^+ - y^+)\delta^{(2)}(x_\perp - y_\perp) \quad (2.43)$$

$$\left\{ \eta_+(x^-, x_\perp), \eta_+^\dagger(y^-, y_\perp) \right\} = \Lambda_+\delta(x^- - y^-)\delta^{(2)}(x_\perp - y_\perp) \quad (2.44)$$

$$[T_i, \phi] = [T_i, \lambda] = [\phi, \phi] = [\lambda, \lambda] = [\eta_+, T] = [\eta_+, \lambda] = \{\eta_+, \eta_+\} = 0 \quad (2.45)$$

fulfill almost all of these requirements. It is straightforward to check that they give the correct Heisenberg equations, except that the commutator of  $\eta_+$  with  $P^-$  fails to reproduce the term containing  $\lambda$  in Eq. (2.41). All the other Heisenberg equations, including the kinematical ones, work correctly.

In order to obtain an interacting light-cone theory that is isomorphic to the theory described in Ref. [4], then, we would need to impose commutation relations more complicated than (2.42)–(2.45). Determining the necessary field algebra would seem to be quite difficult. The required commutation relations would have to correctly give the term proportional to  $\lambda$  in Eq. (2.41), without of course upsetting any of the other Heisenberg equations. The Poincaré algebra must furthermore be satisfied, up to terms proportional to  $\lambda$  (more on this below). The required commutators could be computed perturbatively, for example following the approach of Ref. [13]. One would quantize the theory at equal time and use perturbation theory to solve for the fields everywhere in spacetime. The resulting fields could then be evaluated on the various surfaces of interest and their algebra determined by direct computation. Some work in this direction is in progress. Alternatively one could experiment with further field redefinitions in an effort to obtain fields with simpler commutators.

Let us turn for the moment to the issue of the projection onto a physical subspace. This subspace is defined by the requirement that the equations of motion reduce to Maxwell's equations between physical states. Now, the only equation of motion that is not one of Maxwell's equations is Eq. (2.6), due to the appearance of  $\lambda$  on its right hand side. Therefore  $\lambda$  must have vanishing matrix elements between physical states, or equivalently

$$\lambda^{(+)}|\text{phys}\rangle = 0. \quad (2.46)$$

Note that in this subspace the extra term on the right side of Eq. (2.41) vanishes. Thus the theory defined by the generators we have constructed with the free-field commutation relations (2.42)–(2.45) is equivalent to the original theory on the physical subspace. The resulting theory is just the “naive” light-cone theory tensored with the unphysical fields  $\phi$  and  $\lambda$ , which are decoupled. They can therefore be discarded by invoking the condition (2.46). Clearly, the states in the resulting theory have positive norm so that unitarity holds, and furthermore the Poincaré algebra is satisfied.

Thus while it seems difficult to construct a light-cone version of the theory (2.4) that is isomorphic to the corresponding equal-time theory, on the light cone there is a field redefinition which essentially serves to disentangle the physical and unphysical degrees of freedom. This happens in such a way that simple (free-field) commutation relations give a theory that is equivalent to the full theory in the physical subspace.

### III. THE SCHWINGER MODEL

We shall now show that for the case of the Schwinger model (electrodynamics of massless fermions in two spacetime dimensions), mixing of the gauge field with the Fermi field to form a renormalized current operator modifies the details of the constraint equation for the zero mode fields. The general solution of the Maxwell equation

$$\partial_-^2 A^- = -\frac{1}{2}gJ^+ , \quad (3.1)$$

including the zero modes, is

$$A^- = -g\frac{1}{2\partial_-^2}J^+ + \varphi(x^+)x^- + \gamma(x^+) . \quad (3.2)$$

The main point we wish to make is that for the case of two dimensions we know how the gauge field mixes with the Fermi field to form the current. If we use point-splitting and normal-ordering to define the product of Fermi fields then we find [14]

$$J^- = j^-(x^+) - \frac{g}{2\pi}A^- , \quad (3.3)$$

where

$$j^-(x^+) \equiv \lim_{\epsilon \rightarrow 0} \left\{ \psi_-^\dagger(x^+ + \epsilon)\psi_-(x) - \text{V.E.V.} \right\} . \quad (3.4)$$

That  $j^-$  is a function of  $x^+$  only can be seen from the Dirac equation:

$$\partial_- \psi_- = 0 . \quad (3.5)$$

If we now insert Eq. (3.2) into the other Maxwell equation

$$2\partial_+ \partial_- A^- = 2\lambda + gJ^- \quad (3.6)$$

we see that  $\varphi$  must be zero, since otherwise the RHS of Eq. (3.6) would contain a term linear in  $x^-$  while the LHS would not. We therefore obtain for the constraint

$$\lambda = \frac{g^2}{4\pi}\gamma - \frac{1}{2}gj^- . \quad (3.7)$$

If we define a field  $\sigma$  by the relation

$$\gamma \equiv \frac{4\sqrt{\pi}}{g}\partial_+ \sigma \quad (3.8)$$

we find that consistency between the equations of motion and the Heisenberg equations forces  $\sigma$  to be a ghost field (i.e., it creates and destroys negative-norm states). Furthermore, from the Dirac equation

$$i\partial_+ \psi_+ = \frac{1}{2}gA^- \psi_+ \quad (3.9)$$

we find that we can write

$$\psi_+ = e^{-i2\sqrt{\pi}\sigma}\eta_+, \quad (3.10)$$

which is somewhat like Eq. (2.32), although the details differ.

What we have found here is precisely the structure found in a full light-cone gauge operator solution to the Schwinger model given by Y. Nakawaki [8]. While it may appear to be possible to make a residual ( $x^-$ -independent) gauge transformation which would remove the field  $\sigma$  from both the Fermi field and the gauge field, that possibility does not actually exist. If we make the required (nonlocal) gauge transformation, then the operator products necessary to define the current as a split and gauge-corrected object become undefined and the solution is destroyed. Thus, while the operator mixing changes the details, the situation in the Schwinger model is much like in four dimensional QED: we must retain certain zero mode fields, some of which must be ghosts. In both cases the purpose of these fields is the same: to correct the operator products.

#### IV. DISCUSSION

We have shown how to set up the classical boundary-value problem for the theory (2.4) defined on lightlike surfaces. This includes identifying the independent data, which correspond to operators in the associated quantum field theory, and uncovering constraints that follow from the equations of motion and from requiring finiteness of the classical energy-momentum. The problem we have not solved is that of determining the field algebra that results in the Heisenberg equations reproducing the exact field equations of the theory. There is a field redefinition, however, that results in the almost complete disentangling of the physical and unphysical fields, and allows a theory with simple commutation relations to be defined that is equivalent to the full theory on the physical subspace. This simpler theory is just the naive light-cone QED.

The restriction (2.14) on the current that couples to  $A^-$  appears to be necessary in the (continuum) light cone approach. Its precise meaning, however, is bound up with issues of regularization and renormalization of the current operators, so that without being more specific about these it is impossible to make definite statements.

The extension of this work to the non-Abelian case is at present somewhat unclear. By demanding finiteness of the energy-momentum, we again find that the boundary value of the transverse fields is related to the unphysical field  $\phi^a$  which occurs in the first integral of the light-cone Gauss' law. This boundary value is again a pure gauge, with a gauge function given as an infinite series in the coupling, and serves to motivate a field redefinition to disentangle the physical and unphysical fields. All of this does in fact go through as desired, and what results are expressions for the energy-momentum in which the physical and unphysical fields are decoupled, and equations of motion which are of the usual light-cone form but containing extra terms proportional to  $\lambda^a$  [15]. It therefore appears possible to define a theory which is equivalent to the full theory on the physical subspace, as we have discussed here for QED. The present difficulty is that the boundary contributions to the Poincaré generators differ from their free-field forms, so that they must have complicated commutation relations among themselves in order to obtain the correct Heisenberg equations. We are currently

studying this problem, and hope to report more completely on the non-Abelian case in the near future.

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