Atomic Hardy Space Theory for Unbounded Singular Integrals

RYAN BERNDT

ABSTRACT. We examine singular integrals of the form

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{B(y)}{y} f(x-y)\, dy$$

where the function B is non-negative and even, and is allowed to have singularities at zero and infinity. The operators we consider are not generally bounded on $L^2(\mathbb{R})$, yet there is a Hardy space theory for them. For each T there are associated atomic Hardy spaces, called H^1_B and $H^{1,1}_B$. The atoms of both spaces possess a size condition involving B. The operator T maps $H^{1,1}_B$ and certain H^1_B continuously into $H^1 \subset L^1$. The dual of H^1_B is a space we call BMO_B. The Hilbert transform is a special case of an operator T and its H^1_B and BMO_B spaces are H^1 and BMO.

1. INTRODUCTION

We discuss a Hardy space theory for a certain class of singular integral operators which are, in general, unbounded on $L^2(\mathbb{R})$. By "Hardy space theory" we mean a theory that substitutes for the operators' lack of continuity on $L^1(\mathbb{R})$.

The operators in which we are interested are of the form

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{B(y)}{y} f(x-y)\, dy, \tag{1.1}$$

where B is a non-negative function from a class of functions we discuss in a moment. One such function B has singularities like $|\log|y||$ near the origin and infinity—such a function is not allowed in the standard Calderón-Zygmund theory.

Our work in one dimension was prompted by the n dimensional, $n \geq 2$, work of R. Fefferman [5] on singular integral operators possessing additional functions in their kernels. Continuity on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, was obtained for $\text{pv}(K * f)$ where the kernel K was of the form

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n} B(|x|).$$

The function Ω satisfied certain smoothness and cancelation conditions. In this case the function B was bounded. But, such a theorem does not carry-over to the one dimensional case. When $n = 1$, then $K(x)$ is of the form $B(x)/x$ where B is even. If $B(x) = \sin|x|$, then $\text{pv}(K * f)$ is *not* bounded on $L^2(\mathbb{R})$ [8].

We have found a condition for B other than boundedness that has proven useful for a discussion of the properties of the associated operator T. Let $\mathcal{B}$ be the set of all positive, even functions B for which there exists a constant c_0, $0 \leq c_0 < 1$, such that

$$|B'(x)| \leq c_0 \frac{B(x)}{|x|} \tag{1.2}$$

for all $x \neq 0$. The set $\mathcal{B}$ enjoys many closure properties. For example, the composition of two functions in $\mathcal{B}$ is an element of $\mathcal{B}$. Other properties of $\mathcal{B}$ are discussed in Section 2. We note that if $B(x) = 1/\pi$, then B satisfies (1.2) and $B(x)/x$ is the kernel of the Hilbert transform.

If $B \in \mathcal{B}$, then we may have singularities in the kernel other than $1/x$. For example, if $0 < \alpha < 1$ we may take $B(x) = |x|^\alpha$. In this case $c_0 = \alpha$ suffices to satisfy (1.2). We might call $B(x)/x$ an odd "Riesz potential." Another example is $B(x) = \sqrt{1 + \log^2|x|}$. More examples of functions satisfying (1.2) are discussed in Section 2.

The operators given in (1.1) with $B \in \mathcal{B}$ are not necessarily bounded on $L^2(\mathbb{R})$. This is because of the following theorem.

Theorem 1.1 ([2]). *If $B \in \mathcal{B}$ and $K(x) = B(x)/x$, then $\hat{K}$ exists as a tempered distribution and can be identified with a function (which we also call $\hat{K}$) such that for $z \in \mathbb{R}$,*

$$\hat{K}(z) = -i\,\text{sgn}(z) B(\pi/z) s(z).$$

The function s is a real function that is bounded above and below by positive constants.

We remark that Theorem 1.1 is a generalization of the the identity

$$\hat{h}(z) = -i\,\text{sgn}(z)$$

where $h(x) = 1/\pi x$, the kernel of the Hilbert transform.

The immediate consequence of Theorem 1.1 is that when $B \in \mathcal{B}$, the operator T is bounded on $L^2(\mathbb{R})$ *if* and *only if* B is also bounded. We will consider the case where B is bounded (and so when T is bounded on L^2), but we are particular interested in the case where B is unbounded and hence when T is unbounded on $L^2(\mathbb{R})$.

This brings us to the main point of the paper: finding atomic spaces that non-Calderón-Zygmund singular integral operators map continuously into L^1. We do this even in the case where the operator is unbounded on $L^2(\mathbb{R})$. Since the proof that the Hilbert transform maps H^1 continuously into L^1 usually relies on the fact that the Hilbert transform is bounded on L^2, substitute methods must be developed. Thus, another part of this paper's goal is to show that boundedness on L^2 is not essential for the singular integral operator to have important mapping properties.

We make a brief note about notation. The letter B will always denote a function from the class $\mathcal{B}$ and $K(x)$ will always equal $B(x)/x$. Thus, K will always be an odd function. The letter T will signify the operator $Tf(x) = \text{pv}(K * f)(x)$. When $B(x) = \pi^{-1}$, then T is the Hilbert transform and we use H to denote this operator. We write f_Q for the mean value of a function on the interval Q. Finally, we use the letter C to denote a constant that is suitably independent of the other quantities appearing in context. The value of C may vary at each appearance.

2. Nature of the Kernel and Examples

Definition 2.1. The class of functions $\mathcal{B}$ consists of non-negative, even functions that are positive except perhaps at zero. Each $B \in \mathcal{B}$ is differentiable away from 0 and has a constant c_0, $0 \le c_0 < 1$, associated with it so that

$$|B'(x)| \le c_0 \frac{B(x)}{|x|} \quad \text{for all } x \ne 0. \tag{2.1}$$

We note that if $B(x) = 1/\pi$, then $B \in \mathcal{B}$ with $c_0 = 0$ and $K(x) = B(x)/x$ is the kernel of the Hilbert transform.

The class $\mathcal{B}$ is closed under addition and composition and partially closed under multiplication. We list some closure properties below. The reader may readily verify them.

(i) $\mathcal{B}$ is closed under addition.
(ii) If $B_1, B_2 \in \mathcal{B}$ with associated constants c_0 and c_0', then $B_1 B_2 \in \mathcal{B}$ provided $c_0 + c_0' < 1$.
(iii) If $0 \le \theta \le 1$ and $B_1, B_2 \in \mathcal{B}$, then $B_1^{\theta} B_2^{1-\theta} \in \mathcal{B}$.
(iv) If $B_1, B_2 \in \mathcal{B}$, then $B_1 \circ B_2 \in \mathcal{B}$.
(v) If $B \in \mathcal{B}$, then $1/B \in \mathcal{B}$.
(vi) If $B(x) \in \mathcal{B}$, then $B(1/x) \in \mathcal{B}$.

The functions in $\mathcal{B}$ enjoy other useful properties as a consequence of (2.1). It implies $B(x)$ is integrable near zero and $B(x)/x^2$ is integrable at infinity, see Lemma 2.5. These two statements are in fact equivalent because of (vi) above.

Second, $B(x)/x$ is a decreasing function for $x > 0$ and $xB(x)$ is increasing for $x > 0$, see Lemma 2.2.

It is sometimes more useful to express (2.1) in terms of the function $K(x) = B(x)/x$. If $K(x) = B(x)/x$ and $B \in \mathcal{B}$, then there exist constants c_1 and c_2 with $0 < c_1 \leq c_2 < 2$ such that for $x \neq 0$,

$$c_1 \frac{K(x)}{x} \leq -K'(x) \leq c_2 \frac{K(x)}{x}. \tag{2.2}$$

The inequality (2.2) is equivalent to (2.1) and we will use the characterization (2.2) when it is convenient, see Lemma 2.4. Since $K(x) = B(x)/x$, K is an odd function, differentiable for $x \neq 0$, and decreasing for positive x.

Before we prove these assertions about the functions $B \in \mathcal{B}$, we present some concrete examples of kernels K that satisfy (2.2). For example

$$K(x) = \frac{\sqrt{1 + \log^2 |x|}}{x}$$

is an example of a kernel satisfying (2.2) We note that near 0 and ∞, $K(x) \sim |\log x|/x$. Here, we may take $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

Another example is given by an odd version of the Riesz potentials. That is, for $0 < \alpha < 1$ we may take $K(x) = x^{-1+\alpha}$ for $x > 0$ and extend K to be odd. In this case we have $c_1 = c_2 = 1 - \alpha$ in (2.2).

The kernel $K(x)$ may be *more* or *less* singular at zero and infinity than is allowed in the classical theory. For example, we may take $0 < \beta < \frac{1}{2}$ and we may take $K(x)$ to be the odd extension of any of the following functions:

$$\frac{1}{x^{1-\beta}\sqrt{1 + \log^2 |x|}}, \quad \frac{1}{x^{1+\beta}\sqrt{1 + \log^2 |x|}}, \quad \frac{\sqrt{1 + \log^2 |x|}}{x^{1+\beta}}, \quad \frac{\sqrt{1 + \log^2 |x|}}{x^{1-\beta}}.$$

Even more striking is the following example. Since $|x|^{1/2} \in \mathcal{B}$ we have $|x|^{-1/2} \in \mathcal{B}$ and hence $|x|^{1/2} + |x|^{-1/2} \in \mathcal{B}$. Thus, the odd extension of

$$K(x) = x^{-1/2} + x^{-3/2}$$

satisfies (2.2). Comparing to $1/x$, this kernel is more singular at zero *and* at infinity.

Finally, the kernel $K(x)$ may have an oscillating factor in the numerator (although it must remain positive). For example, for any c_0 with $0 \leq c_0 < 1$,

$$K(x) = \frac{\exp\left(c_0 \int_0^x \left(\frac{\sin t}{t}\right) dt\right)}{x}$$

satisfies (2.2).

We will use the following lemmas in Section 3. The reader may wish to postpone reading them until they are needed in Section 3.

Lemma 2.2. *If $B \in \mathcal{B}$, then for positive x, $B(x)x$ is strictly increasing and $B(x)/x$ is strictly decreasing.*

Proof. This is easily checked by showing that the derivative of $B(x)x$ is positive and the derivative of $B(x)/x$ is negative. □

Lemma 2.3 relates with an inequality the behavior of B under dilation with the functions x^{c_0} and x^{-c_0}. It also provides a representation formula for B.

Lemma 2.3. *If $B \in \mathcal{B}$ with constant c_0, then there exists a bounded function $s(x)$, $|s(x)| \le c_0$, such that*

$$B(x) = B(1)e^{\int_1^x (s(t)/t)\,dt} \tag{2.3}$$

for a.e. $x > 0$. Hence, if $\lambda \ge 1$, then $\lambda^{-c_0}B(x) \le B(\lambda x) \le \lambda^{c_0}B(x)$. In particular $K(x) \to 0$ as $x \to \infty$.

Proof. Take $x > 0$. We may write (2.1) as

$$\frac{B'(x)}{B(x)} = \frac{s(x)}{x}$$

where $|s(x)| \le c_0$. In Lemma 2.2 we saw that $B(x)/x$ is decreasing for $x > 0$ so for any compact set $K \subset (0,\infty)$, $|B'(x)| \le c_0\, B(x)/x \le c_0 C_K$ for some constant C_K depending on K. Thus, $B'(x)$ is locally bounded away from zero and hence B is locally absolutely continuous away from zero. In particular, $\log B(x)$ can be recovered from the integral of its derivative for almost every x in any closed interval contained in $(0,\infty)$. Therefore, we may solve the differential equation and get (2.3). □

Often we will apply a weaker version of the dilation property given in Lemma 2.3, namely if $\lambda \ge 1$, then

$$\lambda^{-1}B(x) \le B(\lambda x) \le \lambda B(x). \tag{2.4}$$

Lemma 2.4. *Let $K(x) = B(x)/x$ where B is an even function.*

(i) *If there exist constants c_1 and c_2, $0 < c_1 \le c_2 < 2$, such that for all $x \ne 0$*

$$c_1\frac{K(x)}{x} \le -K'(x) \le c_2\frac{K(x)}{x},$$

then $B \in \mathcal{B}$ with $c_0 = \max\{1 - c_1, c_2 - 1\}$. Furthermore, if $c_2 < 1$, then B is strictly increasing for $x > 0$ and if $c_1 > 1$, then B is strictly decreasing for $x > 0$.

(ii) *If $B \in \mathcal{B}$, then there exist constants c_1 and c_2, $0 < c_1 \le c_2 < 2$, such that for $x \neq 0$*

$$c_1 \frac{K(x)}{x} \le -K'(x) \le c_2 \frac{K(x)}{x}$$

where $1 - c_2 = \inf_{x>0} B'(x)x/B(x)$ and $1 - c_1 = \sup_{x>0} B'(x)x/B(x)$.

Proof. We show (i) for $x > 0$. Since $-K'(x) = B(x)/x^2 - B'(x)/x$ and $K(x) = B(x)/x$, we have

$$c_1 \frac{B(x)}{x^2} \le \frac{B(x)}{x^2} - \frac{B'(x)}{x} \le c_2 \frac{B(x)}{x^2}, \tag{2.5}$$

which is equivalent to

$$(1 - c_1)\frac{B(x)}{x} \ge B'(x) \ge (1 - c_2)\frac{B(x)}{x}.$$

Thus, if $c_2 < 1$, then B is strictly increasing for positive x and we may take $c_0 = 1 - c_1$. If $c_1 > 1$, then B is strictly decreasing for positive x and we may take $c_0 = c_2 - 1$. Otherwise, $0 < c_1 \le 1 \le c_2 < 2$ and so $|B'(x)| \le \max(1 - c_1, c_2 - 1)B(x)/x$. We note that in all cases, we may actually take $c_0 = \max(1-c_1, c_2-1)$.

We now prove (ii). We take $x > 0$ and write (2.1) as

$$-c_0 \le \frac{B'(x)x}{B(x)} \le c_0.$$

Then we have

$$-c_0 \le \inf_{x>0} \frac{B'(x)x}{B(x)} \le \frac{B'(x)x}{B(x)} \le \sup_{x>0} \frac{B'(x)x}{B(x)} \le c_0.$$

Taking $1 - c_2 = \inf_{x>0} B'(x)x/B(x)$ and $1 - c_1 = \sup_{x>0} B'(x)x/B(x)$ we have

$$c_2 \ge 1 - \frac{B'(x)x}{B(x)} \ge c_1$$

where $0 < 1 - c_0 \le c_1 \le c_2 \le 1 + c_0 < 2$. We multiply the inequality by $B(x)/x^2$ to get

$$c_2 \frac{B(x)}{x^2} \ge \frac{B(x)}{x^2} - \frac{B'(x)}{x} \ge c_1 \frac{B(x)}{x^2}.$$

This is exactly (2.5) and so (ii) is proven. □

Lemma 2.5. *If $B \in \mathcal{B}$, then $B(x)$ is integrable near $x = 0$ and $B(x)/x^2$ is integrable near $x = \infty$. In particular, we have the estimates*

$$\frac{1}{2 - c_1} sB(s) \le \int_0^s B(x)\,dx \le \frac{1}{2 - c_2} sB(s) \tag{2.6}$$

$$\frac{1}{c_2}\frac{B(s)}{s} \le \int_s^\infty \frac{B(x)}{x^2}\,dx \le \frac{1}{c_1}\frac{B(s)}{s}. \tag{2.7}$$

Proof. In Lemma 2.3 we saw that B is locally absolutely continuous away from zero since B is differentiable and B' is locally bounded. Similarly, K is locally absolutely continuous away from zero. Let $\varepsilon > 0$. Using the identity $B(x) = xK(x)$ and integrating by parts we have

$$\int_\varepsilon^s xK(x)\,dx = -\frac{1}{2}\int_\varepsilon^s x^2K'(x)\,dx + \frac{1}{2}s^2K(s) - \frac{1}{2}\varepsilon^2K(\varepsilon). \tag{2.8}$$

Since $-K'(x) \le c_2K(x)/x$, $s^2K(s) = sB(s)$ and $\varepsilon^2K(\varepsilon)$ is positive we have

$$2\int_\varepsilon^s xK(x)\,dx \le c_2\int_\varepsilon^s xK(x)\,dx + sB(s).$$

So that

$$\int_\varepsilon^s xK(x)\,dx \le \frac{1}{2-c_2}sB(s).$$

Fatou's lemma gives us the right hand inequality of (2.6). To get the other half of (2.6) we again integrate by parts to get (2.8). Then, we apply the inequality $-K'(x) \ge c_1K(x)/x$ to get

$$2\int_\varepsilon^s xK(x)\,dx \ge c_1\int_\varepsilon^s xK(x)\,dx + s^2K(s) - \varepsilon^2K(\varepsilon). \tag{2.9}$$

We claim that $\varepsilon^2K(\varepsilon) \to 0$ as $\varepsilon \to 0$. In Lemma 2.3 we saw that $B(x)/x \to 0$ as $x \to \infty$ for any function $B \in \mathcal{B}$. Since $B(1/x)$ is also an element of $\mathcal{B}$ we know that $B(1/x)/x \to 0$ as $x \to \infty$. Then, by substituting ε for $1/x$ we find that $\varepsilon^2K(\varepsilon) = \varepsilon B(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, after taking the limit of both sides of (2.9) and applying the dominated convergence theorem we obtain

$$2\int_0^s xK(x)\,dx \ge c_1\int_0^s xK(x)\,dx + s^2K(s).$$

Thus,

$$\int_0^s xK(x)\,dx \ge \frac{1}{2-c_1}sB(s).$$

To prove (2.7) we take $R > s$ and apply (2.2) to get

$$\frac{1}{c_2}\int_s^R -K'(x)\,dx \le \int_s^R \frac{K(x)}{x}\,dx \le \frac{1}{c_1}\int_s^R -K'(x)\,dx$$

and so

$$c_2^{-1}[K(s) - K(R)] \le \int_s^R \frac{K(x)}{x}\,dx \le c_1^{-1}[K(s) - K(R)].$$

Finally, by Lemma 2.3, $K(R) \to 0$ as $R \to \infty$, so taking the limit as $R \to \infty$ gives us

$$\frac{1}{c_2}\frac{B(s)}{s} \le \lim_{R\to\infty}\int_s^R \frac{K(x)}{x}\,dx \le \frac{1}{c_1}\frac{B(s)}{s},$$

and the monotone convergence theorem allows us to pass the limit inside the integral. ◻

3. Mapping Properties of T and an Associated Dual

The Hardy space H^1 can be characterized atomically: any function $f \in H^1$ is a sum of the form $\sum \lambda_k a_k$ where λ_k is a sequence of numbers such that $\sum |\lambda_k| < \infty$, and each a_k is an H^1-atom. An atom in H^1 is a function a with mean value zero supported in an interval Q, satisfying a size condition like $\|a\|_2 \le |Q|^{-1/2}$. If H is the Hilbert transform it is known that $\|Ha\|_1 \le C$ and from this we may deduce that H maps H^1 into L^1. Also, the dual of H^1 is BMO. In this way the Hilbert transform is naturally related to the spaces H^1 and BMO. The work of this section is modeled on the Hilbert transform, where we will obtain a Hardy space and a dual space related to the operator T.

Recall that T is given by

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon} \frac{B(y)}{y} f(x-y)\,dy.$$

We are looking for atomic spaces that T maps into L^1. The elements of these atomic spaces are of the form $\sum \lambda_k a_k$ with $\sum |\lambda_k| < \infty$, like H^1. The atoms have different size conditions on their L^2 norm depending on the situation. The space H^1_B is characterized by atoms where $\|a\|_2 \le C|Q|^{-1/2}B(|Q|)^{-1}$ and $H^{1,1}_B$ atoms are absolutely continuous satisfying $\|a'\|_2 \le C|Q|^{3/2}B(|Q|)$.

Investigating the mapping properties of T on these atomic spaces, we have found an underlying principle at work. First, if the atoms are "smooth," that is absolutely continuous, with a size condition on the derivative of a, then T maps $H^{1,1}_B$ into L^1 continuously, *for all* $B \in \mathcal{B}$. Without a smoothness assumption on the atoms, the nature of the kernel's singularity at zero is fundamental. If B is bounded and $B \in \mathcal{B}$ so that K has a singularity like $1/x$ at the origin, then T is bounded from H^1 into L^1. If the associated constant c_2 from Lemma 2.4 is less than one, then T maps H^1_B into L^1 continuously.

Finally, for *all* $B \in \mathcal{B}$ we find that the Hilbert transform is bounded on H^1_B and $H^{1,1}_B$. Given this we are able to improve the mapping properties of T and conclude that T actually maps H^1_B and $H^{1,1}_B$ into H^1 continuously.

We show that T maps the spaces H^1_B and $H^{1,1}_B$ continuously into L^1 by checking that $\|Ta\|_1 \le C$ for some constant independent of the atom a. M. Bownik [3] has shown that one must exercise caution when deducing the continuity of an operator from its action on atoms. Theorem 3.2 below allows us to avoid the pitfalls that M. Bownik warns us about [3, p. 3540–3541].

Definition 3.1. We call a function $a \in L^2(\mathbb{R})$ an atom if there exists an interval Q such that

(i) Support $a \subset Q$,

(ii) $\int a = 0$.

Theorem 3.2. *Let A be a class of atoms for which there exist an operator T and a constant C such that*

$$\|Ta\|_1 \leq C$$

for all $a \in A$. Let X be a set whose elements are pairs of sequences $(\lambda_k, a_k)_{k=1}^{\infty}$ where $\lambda_k \in \mathbb{C}$ and $\sum |\lambda_k| < \infty$, and $a_k \in A$. Then, if we define an equivalence class on X such that two elements (λ_k, a_k) and (μ_k, e_k) are equivalent if and only if

$$\sum \lambda_k T a_k = \sum \mu_k T e_k, \tag{3.1}$$

then X is a Banach space. We write the elements of X as formal sums $\sum \lambda_k a_k$, where (λ_k, a_k) is any representative element of its equivalence class. The norm of an element $f \in X$ is given by

$$\|f\|_X = \inf\Big\{ \sum |\lambda_k| : \sum \lambda_k a_k \in [f] \Big\}.$$

By the definition of the equivalence relation, T is automatically well defined and one-to-one on X, if we define the action of T on X by

$$T\Big(\sum \lambda_k a_k \Big) = \sum \lambda_k T a_k.$$

Proof. The sums in (3.1) converge in L^1 so there is no question about their existence. Also, since $\sum |\lambda_k| \, \|Ta_k\|_1$ converges, any rearrangement of $\sum \lambda_k a_k$ will be in the same equivalence class. For the time being we write an equivalence class in X as $[f]$ where f is a representative formal sum from the class. We make X into a vector space by defining

Multiplication by a scalar: $\alpha[\sum \lambda_k a_k] := [\sum (\alpha\lambda_k) a_k]$,
Addition: $[\sum \lambda_k a_k] + [\sum \mu_k e_k] := [\sum (\lambda\mu)_k (ae)_k]$,

where $(\lambda\mu)_{2k} = \lambda_k$, $(\lambda\mu)_{2k+1} = \mu_k$, $(ae)_{2k} = a_k$, and $(ae)_{2k+1} = e_k$. With these definitions and our criterion for equivalence the axioms for a vector space are satisfied.

We check that the norm as defined is indeed a norm:

(a) Suppose $\|f\|_X = 0$. Then, there exists a sequence of formal sums in $[f]$ with ℓ^1 coefficients tending to zero. We take $f_n \in [f]$ such that $f_n = \sum_k \lambda_{nk} a_{nk}$ and $\sum_k |\lambda_{nk}| \to 0$ as $n \to \infty$. Thus, $0 \leq \int |Tf_n(x)| \, dx \leq C \sum_k |\lambda_{nk}| \to 0$ as $n \to \infty$. Therefore, by Fatou's Lemma, $Tf(x) = 0$ almost everywhere. Since T is one-to-one, f must be in the zero class of X.

(b) Let $\alpha \in \mathbb{C}$. Then $\|\alpha f\|_X = \inf\{\sum |\alpha|\,|\lambda_k| : \sum_k \lambda_k a_k \in [f]\} = |\alpha|\,\|f\|_X$.
(c) Let $\varepsilon > 0$ be given and choose $\sum \lambda_k a_k \in [f]$ and $\sum \mu_k e_k \in [g]$ such that $\sum |\lambda_k| < \varepsilon + \|f\|_X$ and $\sum |\mu_k| < \varepsilon + \|g\|_X$. Then, $\sum (\lambda\mu)_k (ae)_k \in [f+g]$. Hence,

$$\|f+g\|_X \le \sum |(\lambda\mu)_k| \le \sum |\lambda_k| + \sum |\mu_k| \le \|f\|_X + \|g\|_X + 2\varepsilon.$$

Therefore, X is a normed vector space. It is also a complete. This can be proven in much the same way that one sometimes proves that L^p is complete, see Rudin [10, p. 67], for example. We give an abbreviated proof.

Suppose f_n is a Cauchy sequence in X. Choose a subsequence f_{n_k} whose consecutive terms grow closer and closer together, that is $\|f_{n_{k+1}} - f_{n_k}\|_X \le 2^{-k}$. Let

$$f = f_{n_1} + \sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}.$$

Since the differences $f_{n_{k+1}} - f_{n_k}$ are in X, we may choose numbers λ_{kj} and atoms a_{kj} in A such that $f_{n_{k+1}} - f_{n_k} = \sum_{j=1}^{\infty} \lambda_{kj} a_{kj}$ and

$$\sum_{j=1}^{\infty} |\lambda_{kj}| \le \|f_{n_{k+1}} - f_{n_k}\|_X + \frac{1}{2^k}.$$

Also, $f_{n_1} \in X$ as well so $f_{n_1} = \sum_{i=1}^{\infty} \lambda_i a_i(x)$ where $\sum_{i=1}^{\infty} |\lambda_i| \le C$. Now, $f \in X$ since it has an atomic decomposition given by

$$f = \sum_{i=1}^{\infty} \lambda_i a_i + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{kj} a_{kj},$$

where $\sum_{i=1}^{\infty} |\lambda_i| + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_{kj}| \le C + 2$. It is now routine to show that $f_{n_k} \to f$ in the X norm, and since a Cauchy sequence with a convergent subsequence converges, X is complete. □

Definition 3.3. Let $B \in \mathcal{B}$. If an atom a is supported in an interval with radius b, then we call it a B-atom if

$$\|a\|_2 \le \frac{1}{b^{1/2} B(b)}.$$

If $k \ge 1$ and if $a^{(k-1)}$ is absolutely continuous and

$$\|a^{(k)}\|_2 \le \frac{1}{b^{k+1/2} B(b)},$$

then we call a a smooth B-atom of order k. We note that smooth B-atoms are B-atoms (except for an unimportant multiplicative constant) because $\|a\|_2 \le cb\|a'\|_2$. Also, when B is constant, then B-atoms are H^1 atoms.

Definition 3.4. The Banach space constructed from B-atoms via (3.1) is called H^1_B and the Banach space constructed from smooth B-atoms of order k is called $H^{1,k}_B$. Sometimes we write $H^{1,0}_B$ for H^1_B.

C. Zorko [13] has considered an atomic Hardy space like the H^1_B we present here and J. Alvarez [1] has studied Calderón-Zygmund operators acting on this space. The size condition on the atom [13] is given by $\|a\|_2 \le |Q|^{-1/2}B(|Q|)$, where B is not an element of $\mathcal{B}$, rather $B(x)$ is decreasing and $x^{1/2}B(x)$ is increasing for $x > 0$. From this a Banach space is constructed and its dual is BMO_B as described in Definition 3.10. Our construction of H^1_B complements these results because we construct H^1_B in a special case of B strictly increasing, and find its dual to be BMO_B. Thus, H^1_B as we have defined it, or as C. Zorko defines it, has BMO_B as its dual.

We obtain the necessary estimate $\|Ta\| \le C$ needed to apply Theorem 3.2 in Theorem 3.7. Later, we see how the size conditions in Definition 3.3 can be applied to get the mapping properties for T. As we might expect, estimating the L^1 norm of Ta away from the support of the atom a is easier. Remarkably, the fact that the kernel may be more singular at infinity than $1/x$ poses no problems. We have for any atom (no just B-atoms) centered at the origin, the following theorem.

Theorem 3.5. *Let $B \in \mathcal{B}$ and $K(x) = B(x)/x$; then there exists a constant C such that*

$$\|K * a\|_{L^1(\mathbb{R}-2Q)} \le Cb^{1/2}B(b)\|a\|_2$$

for any atom a supported in $Q = [-b, b]$.

Proof. Since a has mean value 0 and since $x \in \mathbb{R} - 2Q$,

$$|K * a(x)| \le \int_Q |K(x-y) - K(x)|\,|a(y)|\,dy.$$

Thus,

$$\int_{\mathbb{R}-2Q} |K * a(x)|\,dx \le \int_Q \int_{\mathbb{R}-2Q} |K(x-y) - K(x)|\,|a(y)|\,dx\,dy.$$

Given a point $x \in \mathbb{R} - 2Q$ and a point $y \in Q$, there is a point $\bar{x}$ between $x - y$ and x such that

$$|K(x-y) - K(x)| = |K'(\bar{x})|\,|y|$$

by the Mean Value Theorem. Since $|x-y| \ge |x| - |y| \ge |x|/2$ and $|x| > |x|/2$, we have that $|\bar{x}| \ge |x|/2$. By (2.2) and since $K(t)/t$ is positive and decreases as $|t|$ grows, $|K'(\bar{x})| = -K'(\bar{x}) \le c_2 K(\bar{x})/\bar{x} \le c_2 K(x/2)/(x/2)$. By (2.4), $K(x/2)/(x/2) \le 8K(x)/x$. Once again, by (2.2), $K(x)/x \le c_1^{-1}[-K'(x)]$. Thus, $|K'(\bar{x})| \le C[-K'(x)]$.

Therefore,

$$\int_{\mathbb{R}-2Q} |K * a(x)|\, dx \le C \int_{-b}^{b} |a(y)|\,|y|\, dy \int_{b}^{\infty} -K'(x)\, dx.$$

Also, $\int_{b}^{\infty} -K'(x)\, dx = K(b) = B(b)/b$, and

$$\int_{-b}^{b} |a(y)|\,|y|\, dy \le Cb^{3/2}\|a\|_2,$$

hence

$$\|K * a\|_{L^1(\mathbb{R}-2Q)} \le Cb^{1/2}B(b)\|a\|_2.$$

□

Estimating the L^1 norm of Ta near the support of a requires a consideration of subclasses of $\mathcal{B}$ or a smoothness assumption on a. Here, the singularity at zero in the kernel has a dramatic affect on the operator's mapping properties.

Theorem 3.6. *Let $B \in \mathcal{B}$ and $K(x) = B(x)/x$ and $Tf = \text{pv}(K * f)$. Also, let a be an atom supported in an interval $Q = [-b, b]$. Then,*

(i) *If B is bounded, then*

$$\|Ta\|_{L^1(2Q)} \le Cb^{1/2}\|a\|_2.$$

(ii) *If $B \in \mathcal{B}$ and $1 - c_2 = \inf_{x>0} B'(x)x/B(x) > 0$, then*

$$\|Ta\|_{L^1(2Q)} \le Cb^{1/2}B(b)\|a\|_2.$$

(iii) *If a is absolutely continuous, then*

$$\|Ta\|_{L^1(2Q)} \le Cb^{3/2}B(b)\|a'\|_2.$$

Proof. (i) Since $B \in \mathcal{B}$ and B is bounded, T is bounded on $L^2(\mathbb{R})$ by Theorem 1.1, so (i) follows by an application of Hölder's inequality.

(ii) Since $B \in \mathcal{B}$ and $c_2 < 1$ in Lemma 2.4 we have

$$\int_0^s K(t)\, dt \le \frac{1}{1-c_2}B(s),$$

following by an integration by parts in the same way (2.6) was derived. Thus,

$$\begin{aligned}\int_{2Q} |K * a(x)|\, dx &\le \int_{3Q} |K(y)| \int |a(x-y)|\, dy \\ &\le C\|a\|_1 \int_0^{3b} K(t)\, dt \\ &\le C\|a\|_1 B(3b) \\ &\le Cb^{1/2}B(b)\|a\|_2,\end{aligned}$$

because by (2.4), $B(3b) \le 3B(b)$.

(iii) Let $x \in 2Q$. Define the truncated kernel $K_\varepsilon = K\chi_{\{y:|y|>\varepsilon\}}$. We write

$$\begin{aligned} K_\varepsilon * a(x) &= \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} K(y)a(x-y)\,dy \\ &= \int_{\varepsilon}^{\infty} K(y)[a(x-y) - a(x+y)]\,dy, \end{aligned}$$

using the fact that K is odd. Since a is supported in $[-b, b]$ and $x \in [-2b, 2b]$, the above integral is zero when $y > 3b$. So we may as well write the integral over $[\varepsilon, 3b]$ and

$$\begin{aligned} K_\varepsilon * a(x) &= \int_{\varepsilon}^{3b} K(y)[a(x-y) - a(x+y)]\,dy \\ &= 2\int_{\varepsilon}^{3b} yK(y)\left(\frac{1}{2y}\int_{x+y}^{x-y} a'(t)\,dt\right)dy. \end{aligned}$$

The term in parentheses is dominated by the Hardy-Littlewood Maximal function of a', $M(a')(x)$. Hence,

$$|K_\varepsilon * a(x)| \le CM(a')(x)\int_{\varepsilon}^{3b} yK(y)\,dy.$$

Since $\|Ma'\|_{L^1(2Q)} \le Cb^{1/2}\|Ma'\|_2 \le Cb^{1/2}\|a'\|_2$ and $yK(y)$ is integrable on $[0, 3b]$ by (2.6), $K_\varepsilon * a(x)$ is bounded by an integrable function independent of ε. Thus, the Dominated Convergence Theorem implies

$$\lim_{\varepsilon\to 0} K_\varepsilon * a(x) = \int_0^{3b} K(y)[a(x-y) - a(x+y)]\,dy \tag{3.2}$$

for $x \in 2Q$. Therefore, $K * a(x)$ exists as a principal value convolution and has the representation (3.2) as a function for $x \in 2Q$. Now, if we apply the estimate for the integral of $yK(y)$ as given in (2.6) along with the standard L^2 estimate for $M(a')$ we get

$$\begin{aligned} \int_{-2b}^{2b} |K * a(x)|\,dx &\le C\int_{-2b}^{2b} M(a')(x)\,dx \int_0^{3b} yK(y)\,dy \\ &\le Cb^{1/2}\|a'\|_2 \int_0^{3b} yK(y)\,dy \\ &\le Cb^{3/2}B(b)\|a'\|_2. \end{aligned}$$

□

Theorem 3.7. *If $B \in \mathcal{B}$ and $K(x) = B(x)/x$ and $Tf = \mathrm{pv}(K * f)$, then we have the following mapping properties for T.*

(i) *If B is bounded, then $T : H^1 \to L^1$ continuously, that is*

$$\|Tf\|_1 \leq C\|f\|_{H^1}.$$

(ii) *If $B \in B$ and $1 - c_2 = \inf_{x>0} B'(x)x/B(x) > 0$, then $T : H^1_B \to L^1$ continuously, that is*

$$\|Tf\|_1 \leq C\|f\|_{H^1_B}.$$

(iii) *In general, $T : H^{1,1}_B \to L^1$ continuously, that is*

$$\|Tf\|_1 \leq C\|f\|_{H^{1,1}_B}.$$

Proof. Let a be an atom supported in $Q = [-b, b]$.

(i) If a is an H^1 atom, then $\|a\|_2 \leq b^{-1/2}$. By Theorems 3.5 and 3.6 and since B is bounded,

$$\begin{aligned}
\|Ta\|_1 &= \|Ta\|_{L^1(\mathbb{R}-2Q)} + \|Ta\|_{L^1(2Q)} \\
&\leq C(b^{1/2}B(b)\|a\|_2 + b^{1/2}\|a\|_2) \\
&\leq C
\end{aligned}$$

where C depends on $\|B\|_\infty$.

(ii) If a is a B-atom, then $\|a\|_2 \leq b^{-1/2}B(b)^{-1}$. By Theorems 3.5 and 3.6 we have

$$\begin{aligned}
\|Ta\|_1 &= \|Ta\|_{L^1(\mathbb{R}-2Q)} + \|Ta\|_{L^1(2Q)} \\
&\leq C(b^{1/2}B(b)\|a\|_2 + b^{1/2}B(b)\|a\|_2) \\
&\leq C.
\end{aligned}$$

(iii) If a is a smooth B-atom of order 1, then $\|a'\|_2 \leq b^{-3/2}B(b)^{-1}$. By Theorems 3.5 and 3.6 we have

$$\begin{aligned}
\|Ta\|_1 &= \|Ta\|_{L^1(\mathbb{R}-2Q)} + \|Ta\|_{L^1(2Q)} \\
&\leq C(b^{1/2}B(b)\|a\|_2 + b^{3/2}B(b)\|a'\|_2) \\
&\leq Cb^{3/2}B(b)\|a'\|_2 \\
&\leq C.
\end{aligned}$$

Now, completing the proof for each of the three cases is the same. We just prove (ii) and the others follow in the same way. Let $f \in H^1_B$ where $f = \sum \lambda_k a_k$ and where $\sum |\lambda_k| \leq 2\|f\|_{H^1_B}$. If the support of a_k is not centered at the origin, then we translate it to the origin. Since convolutions commute with translations and the L^1 norm is invariant under translations, we have $\|Ta_k\|_1 \leq C$. Hence,

$$\|Tf\|_1 \leq \sum |\lambda_k| \, \|Ta_k\|_1 \leq C \sum |\lambda_k| \leq C\|f\|_{H^1_B}. \qquad \square$$

Now we show that the Hilbert transform operates continuously on H^1_B and $H^{1,k}_B$. This will give us the following improvement to Theorem 3.7.

Theorem 3.8. *If $B \in B(x)$ and $K(x) = B(x)/x$ and $Tf = \mathrm{pv}(K * f)$, then if $T : H^1_B \to L^1$ continuously, then $T : H^1_B \to H^1$ continuously, so*

$$\|Tf\|_{H^1} \leq C\|f\|_{H^1_B}.$$

Similarly, T always maps $H^{1,k}_B$ into H^1 continuously, so

$$\|Tf\|_{H^1} \leq C\|f\|_{H^{1,k}_B},$$

for all $k \geq 1$.

Proof. By Theorem 3.9, if $f \in H^{1,k}_B$, then $Hf \in H^{1,k}_B$ for any $k \geq 0$. Thus, under our assumptions $H(Tf) = T(Hf) \in L^1$ and $Tf \in L^1$ hence $Tf \in H^1$. Now, showing the continuity is routine (for the H^1 norm we use the equivalent norm $\|g\|_1 + \|Hg\|_1$). $\square$

Theorem 3.9. *If $B \in \mathcal{B}$ and $f \in H^{1,k}_B$ for some $k \geq 0$, then there exists a constant C such that*

$$\|Hf\|_{H^{1,k}_B} \leq C\|f\|_{H^{1,k}_B}.$$

Proof. We show that if a is a B-atom with smoothness $k \geq 0$, then we may decompose Ha so that

$$Ha(x) = \sum \lambda_j a_j(x) \tag{3.3}$$

where each a_j is a B-atom with smoothness of order k and $\sum |\lambda_j| \leq C$. Once we have shown (3.3), then we may take any $f \in H^{1,k}_B$ where $f = \sum \mu_k e_k$ with $\sum |\mu_k| \leq 2\|f\|_{H^1_B}$ and apply H to it to get $\|Hf\|_{H^{1,k}_B} \leq C\|f\|_{H^{1,k}_B}$.

We now prove (3.3). Without loss of generality we may assume that a is supported in the interval $[-b, b]$. We decompose the kernel of the Hilbert transform via a partition of unity. The partition of unity [6, p. 322] is given by C^∞ functions φ_j, where $j = 0, 1, 2, \ldots$, that satisfy

(i) $\varphi_j(t) \geq 0$
(ii) $\sum_{j=0}^{\infty} \varphi_j(t) = 1$ for all $t \in (0, \infty)$
(iii) φ_0 is supported in $[0, 2b]$

(iv) φ_j is supported in $[2^{j-1}b, 2^{j+1}b]$, for $j = 1, 2, 3, \ldots$

(v) $|\varphi_j^{(k)}(t)| \le d_k t^{-k}$ for all $t > 0$, $k = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$

Let $h(x) = (\pi x)^{-1}$ be the kernel of the Hilbert transform. Let $h_j(x) = h(x)\varphi_j(|x|)$ for $j = 0, 1, 2, \ldots$. Then,

$$|h_j^{(k)}(x)| \le \frac{C}{|x|^{k+1}}. \tag{3.4}$$

Also, $h(x) = h(x)\sum_{j=0}^{\infty}\varphi_j(|x|) = \sum_{j=0}^{\infty} h(x)\varphi_j(|x|) = \sum_{j=0}^{\infty} h_j(x)$. We note that for any x this sum has at most three terms because of the locations of the supports of the functions φ_j. Thus,

$$Ha(x) = h * a(x) = \sum_{j=0}^{\infty} h_j * a(x).$$

Now, $\bar{h} = h_0 + h_1 + h_2$ is supported in $[-8b, 8b]$ and gives rise to a bounded operator on $L^2(\mathbb{R})$ since the Hilbert transform is itself bounded on L^2. Thus, for some constant A,

$$\|\bar{h} * a^{(k)}\|_2 \le A\|a^{(k)}\|_2 \le \frac{A}{b^{k+1/2}B(b)}.$$

The function $\bar{h} * a$ is supported in $[-8b, 8b] + [-b, b] = [-9b, 9b]$ and has mean-value zero. By (2.4), the weak dilation property of the functions B, $B(9b) \le 9B(b)$, so we have

$$\|\bar{h} * a'\|_2 \le \frac{9^{k+3/2}A}{(9b)^{k+1/2}B(9b)}.$$

Hence, we may write $\bar{h} * a(x) = \lambda_0 a_0(x)$ where $\lambda_0 = 9^{k+3/2}A$ and a_0 is a B-atom with smoothness of order k.

We now consider the case $j \ge 3$. Since a has mean-value zero,

$$h_j^{(k)} * a(x) = \int_{-b}^{b} [h_j^{(k)}(x-y) - h_j^{(k)}(x)]a(y)\,dy.$$

By the Mean Value Theorem and (3.4)

$$|h_j^{(k)} * a(x)| \le C\int_{-b}^{b} \frac{|y|}{|t|^{k+2}}|a(y)|\,dy$$

for some t between $x - y$ and y. We may as well assume that x is in the support of $h_j * a$. We have

$$\begin{aligned}\operatorname{supp} h_j * a(x) &\subset \{x : 2^{j-1}b \le |x| \le 2^{j+1}b\} + \{x : 0 \le |x| \le b\}\\ &\subset \{x : 2^{j-1}b - b \le |x| \le 2^{j+1}b + b\}.\end{aligned}$$

Since $y \in [-b, b]$ we have $|x - y| \geq |x| - |y| \geq 2^{j-1}b - 2b \geq 2^{j-2}b$. Also, $|x| \geq 2^{j-2}b$. Thus, $|t| \geq 2^{j-2}b \geq C2^{j}b$. Therefore,

$$
\begin{aligned}
|h_j^{(k)} * a(x)| &\leq \frac{C}{2^{j(k+2)}b^{k+2}} \int_{-b}^{b} |y|\,|a(y)|\,dy \\
&\leq C\frac{b^{3/2}}{2^{j(k+2)}b^{k+2}}\|a\|_2 \\
&\leq \frac{C}{2^{j(k+2)}}\frac{1}{b^{k+1}B(b)}.
\end{aligned}
$$

The support of $h_j * a(x)$ is contained in the interval $[-2^{j+2}b, 2^{j+2}b]$, so we observe

$$
\begin{aligned}
\frac{1}{b^{k+1}B(b)} &= \frac{(2^{j+2}b)^{k+1}B(2^{j+2}b)}{b^{k+1}B(b)}\frac{1}{(2^{j+2}b)^{k+1}B(2^{j+2}b)} \\
&\leq C2^{(j+2)(k+1)}\frac{B(2^{j+2}b)}{B(b)}\frac{1}{(2^{j+2}b)^{k+1}B(2^{j+2}b)} \\
&\leq C2^{(j+2)(k+1)+jc_0}\frac{1}{(2^{j+2}b)^{k+1}B(2^{j+2}b)}.
\end{aligned}
$$

To get the last inequality we applied Lemma 2.3. That is, $B(2^{j+2}b) \leq (2^{j+2})^{c_0}B(b)$ where $c_0 < 1$ is the constant associated to the function B. We now have

$$
\begin{aligned}
|h_j^{(k)} * a(x)| &\leq C\frac{2^{(j+2)(k+1)+jc_0}}{2^{j(k+2)}}\frac{1}{(2^{j+2}b)^{k+1}B(2^{j+2}b)} \\
&\leq \left(\frac{C2^{2k}}{(2^{1-c_0})^{j}}\right)\frac{1}{(2^{j+2}b)^{k+1}B(2^{j+2}b)}.
\end{aligned}
$$

If we let $h_j * a(x) = \lambda_j a_j(x)$, where λ_j is the quantity above in parentheses, then a_j is a B-atom with smoothness of order k. Also, the sum of the coefficients is bounded above by a constant not depending on the atom:

$$
\begin{aligned}
\sum_{j=0}^{\infty}|\lambda_j| &= 9^{k+3/2}A + \sum_{j=3}^{\infty}|\lambda_j| \\
&= 9^{k+3/2}A + C2^{2k}\frac{2^{2(1-c_0)}}{2^{1-c_0}-1}.
\end{aligned}
$$

This proves (3.3) and therefore Theorem 3.9. ◻

To better understand these atomic spaces as they are defined we find the dual of one them, namely H_B^1.

Definition 3.10. The space BMO_B is the set of all locally integrable functions g such that there exists a constant C where

$$\left(\frac{1}{|Q|}\int_Q |g(x)-g_Q|^2\,dx\right)^{1/2} \le CB(|Q|)$$

for all intervals $Q \subset \mathbb{R}$. The smallest constant C such that this holds is called the BMO_B (semi) norm of g and is written $\|g\|_{\mathrm{BMO}_B}$. When $B(x) = 1$, then $\mathrm{BMO}_B = \mathrm{BMO}$.

Theorem 3.11. *The dual of* H^1_B *is* BMO_B.

This theorem is a consequence of Lemma 3.12, and Theorems 3.13 and 3.14 below.

Lemma 3.12. *The set of all elements in* H^1_B *given by finite sums is dense in* H^1_B *under its norm.*

Proof. Let $g \in H^1_B$ where $g = \sum_{k=1}^{\infty} \lambda_k a_k$. Let $g_n = \sum_{k=1}^{n} \lambda_k a_k$ be the sequence of partial sums. We show that g_n converges to g in the H^1_B norm. By definition we know that λ_k forms an ℓ^1 sequence. Hence, given $\varepsilon > 0$ there is an N such that $\sum_{k=n}^{\infty} |\lambda_k| < \varepsilon$ for all $n \ge N$. But, by definition

$$\|g-g_n\|_{H^1_B} = \Big\|\sum_{k=n}^{\infty} \lambda_k\, a_k(x)\Big\|_{H^1_B} \le \sum_{k=n}^{\infty} |\lambda_k|.$$

Thus, g_n converges to g in the H^1_B norm. ∎

Theorem 3.13. *Let* $f \in \mathrm{BMO}_B$. *The linear functional given by*

$$\ell(g) = \sum_{k=1}^{\infty} \lambda_k \int f(x)a_k(x)\,dx = \lim_{n\to\infty} \int f(x) \sum_{k=1}^{n} \lambda_k a_k(x)\,dx,$$

where $g = \sum \lambda_k a_k \in H^1_B$, *is well defined and bounded on* H^1_B. *Also,* $\|\ell\| \le C\|f\|_{\mathrm{BMO}_B}$.

Proof. Suppose $g \in [0]$ the zero class in H^1_B; then $\|g\|_{H^1_B} = 0$. Given $\varepsilon > 0$, there exist numbers μ_k and B-atoms e_k such that $\sum |\mu_k| < \varepsilon$ and $g = \sum \mu_k e_k$. If each e_k is supported in Q_k, then

$$\left|\int f(x)e_k(x)\,dx\right| \le \left(\int_{Q_k} |f(x)-f_{Q_k}|^2\,dx\right)^{1/2} \|e_k\|_2 \le C\|f\|_{\mathrm{BMO}_B}.$$

Thus $|\ell(g)| \le C\|f\|_{\mathrm{BMO}_B} \sum_{k=1}^{\infty} |\mu_k| \le C\varepsilon$ and so $\ell(g) = 0$. This proves that ℓ is well defined. Now, showing that ℓ is bounded on H^1_B is entirely similar. Take $g = \sum \lambda_k a_k \in H^1_B$ where $\sum |\lambda_k| \le C\|g\|_{H^1_B}$. Then, $|\ell(g)| \le C\|g\|_{H^1_B}\|f\|_{\mathrm{BMO}_B}$. ∎

Theorem 3.14. *If $\ell \in (H_B^1)^*$, then there exists a function $f \in \mathrm{BMO}_B$ such that*

$$\ell(g) = \int f(x)g(x)\,dx$$

for all $g \in H_B^1$ which are finite linear combinations of B-atoms. By Lemma 3.12 we may extend ℓ to H_B^1. Also, $\|f\|_{\mathrm{BMO}_B} \le C\|\ell\|$.

Proof. The proof of this theorem follows a standard proof for H^1. A sketch of it can be found in Stein [12, p. 143]. Let $\ell \in (H_B^1)^*$ have norm $\|\ell\|$. Fix a closed interval $Q \subset \mathbb{R}$ with length $2b$. Define $A(Q) = \{g \in L^2 : \operatorname{supp} g \subset Q$, and $\int g = 0\}$. The set $A(Q)$ is a Hilbert space under the L^2 norm. If $g \in A(Q)$, then

$$\left\| \frac{g}{\|g\|_2 b^{1/2}B(b)} \right\| = \frac{1}{b^{1/2}B(b)}.$$

So, g is a multiple of a B-atom and

$$\|g\|_{H_B^1} \le \|g\|_2 b^{1/2} B(b).$$

If we restrict ℓ to $A(Q)$, then $|\ell(g)| \le \|\ell\|\,\|g\|_{H_B^1} \le \|\ell\|\,\|g\|_2 b^{1/2}B(b)$ for all $g \in A(Q)$. Thus, the norm of ℓ on $A(Q)$ is less than or equal to $\|\ell\| b^{1/2}B(b)$. By the Riesz representation theorem, there exists a function $f^Q \in A(Q)$ such that

$$\ell(g) = \int_Q f^Q(x)g(x)\,dx$$

for every $g \in A(Q)$ and where $\|f^Q\|_2 \le \|\ell\| b^{1/2}B(b)$. To each interval Q with length $2b$ there is associated such a function $f^Q \in A(Q)$.

Suppose $Q_1 \subset Q_2$; then $\ell(g) = \int_{Q_1} f^{Q_1}(x)g(x)\,dx = \int_{Q_1} f^{Q_2}(x)g(x)\,dx$ for all $g \in A(Q_1)$. Hence, for any constant c we have

$$\int_{Q_1} [f^{Q_2}(x) - f^{Q_1}(x) - c]g(x)\,dx = 0$$

for all $g \in A(Q_1)$. Since $[f^{Q_2}(x) - f^{Q_1}(x) - (f^{Q_2} - f^{Q_1})_{Q_1}]\chi_{Q_1}(x) \in A(Q_1)$ we must have

$$\int_{Q_1} |f^{Q_2}(x) - f^{Q_1}(x) - (f^{Q_2} - f^{Q_1})_{Q_1}|^2\,dx = 0.$$

Thus, f^{Q_1} and f^{Q_2} differ by a constant on Q_1. In fact, $f^{Q_2}(x) - f^{Q_1}(x) = (f^{Q_2} - f^{Q_1})_{Q_1} = f^{Q_2}_{Q_1}$ since f^{Q_1} has mean-value zero on Q_1.

Let Q_j be an increasing sequence of intervals converging to $\mathbb{R}$. Let f^{Q_j} be the function associated with the interval Q_j. Let $f(x) = f^{Q_1}(x)$ for $x \in Q_1$ and $f(x) = f^{Q_j}(x) - f^{Q_j}_{Q_1}$ for $x \in Q_j$, $j \geq 2$. A careful analysis shows that f is well defined.

Now, let Q be an arbitrary interval and let $g \in A(Q)$ where $|Q| = 2b$. We know $\ell(g) = \int_Q f^Q g$, where $\|f^Q\|_2 \leq \|\ell\| b^{1/2} B(b)$. But, we also know that since $Q \subset Q_j$ for some j, $f^{Q_j}(x) - f^Q(x) = f^{Q_j}_Q$. So,

$$\begin{aligned}\ell(g) &= \int_Q (f^{Q_j}(x) - f^{Q_j}_Q) g(x)\,dx \\ &= \int_Q (f^{Q_j}(x) - f^{Q_j}_{Q_1}) g(x)\,dx \\ &= \int f(x) g(x)\,dx.\end{aligned}$$

Also, since $f(x) - f_Q = f^{Q_j}(x) - f^{Q_j}_{Q_1} - f_Q = f^Q(x) + f^{Q_j}_Q - f^{Q_j}_{Q_1} - f_Q$, and $f_Q = (f^{Q_j}(x) - f^{Q_j}_{Q_1})_Q = f^{Q_j}_Q - f^{Q_j}_{Q_1}$ we have $f(x) - f_Q = f^Q(x)$ and

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2\,dx\right)^{1/2} \leq C\|\ell\| B(|Q|)$$

because $\|f^Q\|_2 \leq \|\ell\| b^{1/2} B(b)$ and $B(b) \leq 2B(2b)$. □

4. Examples of H^1_B and BMO_B

We have seen that, when $B(x) = 1$, H^1_B and BMO_B are the conventional spaces H^1 and BMO. We present some other examples of the spaces H^1_B and BMO_B here, as well as some related spaces that have been studied by other authors.

Let $B(x) = |x|^\alpha$ with $0 < \alpha \leq 1$. For the purpose of this example we write BMO_B as BMO_α and H^1_B as H^1_α. Let $g \in \mathrm{BMO}_\alpha$; then by definition

$$\left(\frac{1}{|Q|}\int |g(x) - g_Q|^2\,dx\right)^{1/2} \leq C|Q|^\alpha. \tag{4.1}$$

By the work of Meyers [7] and Campanato [4], the functions that satisfy (4.1) are known to be the space of Lipshitz functions of order α when we consider two functions that differ by a constant to be equal. Writing this quotient space as $\mathrm{Lip}(\alpha)$, we have

$$\mathrm{BMO}_\alpha = \mathrm{Lip}(\alpha).$$

Any power of p, $1 \leq p < \infty$, in the mean oscillation on the left of (4.1) may be used to characterize $\mathrm{Lip}(\alpha)$. These results may be found in Garcia-Cuerva and Rubio de Francia [6, p. 299], for example.

The functions satisfying (4.1) with $|Q|^\alpha$ replaced with $|Q|^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, define the Morrey space $L^{2,\gamma}(\mathbb{R})$ where $\gamma = 1 - 2\alpha$. See the paper of J. Peetre [9, p. 72].

Regarding the continuity of the elements of BMO_B we mention a result of Spanne [11]. If $B(x)/x$ is decreasing, as is the case for $B \in \mathcal{B}$, and

$$\frac{1}{|Q|}\int |g(x) - g_Q|\,dx \le CB(|Q|), \tag{4.2}$$

then g is necessarily continuous provided $B(x)/x$ is integrable on $[0,1]$. Spanne has also shown that if $B(x)/x$ is not integrable on $[0,1]$, then there exists a function g satisfying (4.2) that is neither bounded nor continuous.

Consider the atomic Hardy spaces H^p where $\frac{1}{2} < p < 1$. A function a is an H^p atom if a is supported in an interval of length $2b$ and

(i) Support $a \subset Q$,
(ii) $\int_Q a = 0$,
(iii) $\|a\|_2 \le b^{1/2-1/p}$.

Then, H^p is defined to be

$$H^p = \Big\{ \sum_{k=1}^{\infty} \lambda_k a_k(x) : \Big(\sum_{k=1}^{\infty} |\lambda_k|^p \Big)^{1/p} < \infty \Big\}.$$

Here, the a_k are H^p atoms and the λ_k are numbers. Now, if $\alpha = 1/p - 1$ and $B(x) = |x|^\alpha$, then B-atoms correspond to H^p atoms. It is known [6, p. 307] that $(H^p)^* = \mathrm{Lip}(\alpha)$ and by Theorem 3.11 the dual space of H^1_α is BMO_α, which in turn equals $\mathrm{Lip}(\alpha)$. Therefore,

$$(H^1_\alpha)^* = \mathrm{Lip}(\alpha) = (H^p)^*$$

for $\frac{1}{2} < p < 1$ and $\alpha = 1/p - 1$. However, H^1_α is not equal to H^p, because H^p is not complete whereas H^1_α is complete.

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Department of Mathematics
The Ohio State University

CURRENT ADDRESS:
Department of Mathematics
Kansas State University
138 Cardwell Hall
Manhattan, KS 66506
E-MAIL: berndt@math.ksu.edu

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