2007

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Contemporary Mathematics
Volume **428**, 2007

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Abstract. We formulate a Rubio de Francia type extrapolation theorem on domains Ω in $\mathbf{R}^n$. The machinery required is a pair of local A_p weight classes that give rise to an appropriately defined maximal operator bounded on weighted L^p spaces. The definitions we give of the local A_p classes respect the shape of the boundary of Ω and, in addition, one class of weights can be easily seen to contain superharmonic functions.

Let $w \geq 0$ and let $L^p(w)$ be the weighted L^p class given by measurable functions f such that $\int |f|^p w < \infty$. Roughly speaking, the following celebrated theorem of Rubio de Francia allows us to determine an operator's boundedness on a range of $L^p(w)$ classes given the operator's boundedness on a single $L^r(w)$ class. In this way, the theorem is an *extrapolation* theorem.

Theorem [**1**] and [**4**]. *Let $1 \leq r < \infty$. If T is a bounded operator on $L^r(w)$ for all $w \in A_r(\mathbf{R}^n)$, then T is a bounded operator on $L^p(w)$ for all $w \in A_p(\mathbf{R}^n)$ and for all p, $1 < p < \infty$.*

The class of weights mentioned in the theorem are the well known $A_p(\mathbf{R}^n)$ weights. For the purposes of presentation, we only discuss the case $p > 1$ for the time being. A function $w \geq 0$ defined on $\mathbf{R}^n$ is an A_p weight if and only if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-p'/p}\,dx\right)^{p/p'} \leq C$$

for all $Q \subset \mathbf{R}^n$. As usual, p' is the conjugate exponent to p, so $p' = p/(p-1)$. If M is the (uncentered) Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{Q, x\in Q} \frac{1}{|Q|}\int_Q |f|$$

2000 *Mathematics Subject Classification.* Primary 46E30; Secondary 42B25, 47B38.
Key words and phrases. A_p weights, Muckenhoupt weights, extrapolation.

then $w \in A_p(\mathbf{R}^n)$ if and only if M is bounded on $L^p(w)$. That is,

$$\int_{\mathbf{R}^n} Mf(x)^p w(x)\,dx \le C \int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx.$$

It is our goal to formulate a Rubio de Francia type extrapolation theorem on domains Ω (bounded, open, and connected sets) in $\mathbf{R}^n$. The difficulty lies in finding a definition of local weights and a local maximal function whose relationship is like that in the case $\Omega = \mathbf{R}^n$.

One option is to "cut" the weights at the boundary of Ω. Garcia-Cuerva and Rubio de Francia [**2**, p. 438] record the theorems of T. Wolff who defined a class of local A_p weights designed to solve an extension problem. For $w(x) \ge 0$ defined on Ω, $w \in A_{p,\Omega}$ $(1 < p < \infty)$ if and only if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_{Q\cap\Omega} w(x)\,dx\right)\left(\frac{1}{|Q|}\int_{Q\cap\Omega} w(x)^{-p'/p}\,dx\right)^{p/p'} \le C$$

for all $Q \subset \mathbf{R}^n$. The maximal operator corresponding to these weights is given by

$$N_\Omega f(x) = \sup_{Q,x\in Q} \frac{1}{|Q|}\int_{Q\cap\Omega} |f(y)|\,dy.$$

With these definitions then we have the following theorem.

Theorem [**2**, p. 438—9]. *The function $w \in A_{p,\Omega}$ if and only if N_Ω is weak-type (p,p). Additionally, an element w of $A_{p,\Omega}$ can be extended to all of $\mathbf{R}^n$ so that its extension is an element of $A_p(\mathbf{R}^n)$ if and only if $w^{1+\varepsilon} \in A_{p,\Omega}$ for some $\varepsilon > 0$.*

However, the extension problem and the problem of finding an extrapolation theorem on domains seem to be mutually exclusive. It is well known that for operators defined on domains, the nature of the boundary has a crucial affect on the L^p boundedness of the operator. The problem with the weights $A_{p,\Omega}$ is that they *do not respect the shape of the boundary* and hence they are not the "right" local weights to solve our problem of extrapolation of operators defined on domains.

We present local A_p weights that allow us to solve the extrapolation problem on domains. Our definitions of local A_p weights mesh well with the current theory so that the known theory, as in Garcia-Cuerva and Rubio de Francia [2] for example, can be readily adapted to this setting. Therefore, we provide several examples and give abbreviated proofs.

We will use the conventions that $w(Q) = \int_Q w$ and $w_Q = w(Q)/|Q|$. Also, we frequently use σ for the function $w^{-p'/p}$. As in the theory of $A_p(\mathbf{R}^n)$ weights it is sometimes more convenient to use balls instead of cubes; one can replace the cubes Q below with balls and all the definitions will make sense and all the results will remain valid.

(1) Definition. A small interior cube Q of Ω is a cube Q such that $3Q \subset \Omega$. Here, $3Q$ is the three times concentric dilation of Q.

The well known "Whitney" cubes are examples of small interior cubes. The difference between the two classes is that although small interior cubes can not be too large near the boundary, they are allowed to be very small in general.

Let $d(x)$ be the distance from x to the boundary of Ω, $b\Omega$. If Q is a small interior cube then $3Q \subset \Omega$ by definition. Since $|Q| > 0$ and $|\Omega| < \infty$ there exists

a $k \in \mathbf{N}$ such that $3^k Q \subset \Omega$ but $3^{k+1} Q \not\subset \Omega$. As a consequence if $\ell(Q)$ is the sidelength of Q then

$$c_1 3^k \ell(Q) < d(x) < c_2 3^{k+1} \ell(Q), \tag{2}$$

where c_1 and c_2 are dimensional constants.

(3) Definition. For $1 < p < \infty$, we say that the A_p inequality holds for a collection of cubes $\mathcal{Q}$ if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-p'/p}\,dx\right)^{p/p'} \leq C$$

When $p = 1$ we say that w satisfies the A_1 inequality for a collection of cubes $\mathcal{Q}$ if there exists a constant C such that

$$\frac{1}{|Q|}\int_Q w(t)\,dt \leq Cw(x)$$

for all cubes in $Q \in \mathcal{Q}$ and for almost every $x \in Q$.

(4) Definition. A nonnegative function $w \in A_p(\Omega)$ if and only if the A_p inequality holds for all $Q \subset \Omega$.

(5) Definition. A nonnegative function w is an element of *weak* $A_p(\Omega)$ if and only if the A_p inequality holds for all small interior cubes in Ω.

We remark that $A_p(\Omega) \subset$ *weak* $A_p(\Omega)$ and that both classes $A_p(\Omega)$ and *weak* $A_p(\Omega)$ grow as $p \geq 1$ increases. Also, these two classes are certainly not equal. For example, if Ω is the open interval $(0,1)$ on the real line then $w(x) = 1/x$ is an element of *weak* $A_2(\Omega)$ but not an element of $A_2(\Omega)$.

The following lemma shows the relationship between $A_p(\Omega)$ and *weak* $A_p(\Omega)$. We say that a nonnegative function is doubling on a collection of cubes if there exists a constant C such that $\int_{3Q} w \leq C \int_Q w$ for all cubes in the collection.

(6) Lemma. *Let $p > 1$. A function $w \in A_p(\Omega)$ if and only if $w \in$ weak $A_p(\Omega)$ and w and $w^{-p'/p}$ are doubling on all small interior cubes.*

Proof. The only thing not immediately clear is that $w \in A_p(\Omega)$ implies that w and $w^{-p'/p}$ are doubling on small interior cubes. We prove the contrapositive. Suppose that w is not doubling on small interior cubes. Then, for every number $N > 0$ there exists a small interior cube Q such that $w(3Q) > Nw(Q)$. With $\sigma = w^{-p'/p}$ we have by Hölder's inequality,

$$w(3Q)\sigma(3Q)^{p/p'} \geq Nw(Q)\sigma(Q)^{p/p'} \geq N|Q|^p.$$

Since Q is a small interior cube, $3Q \subset \Omega$, and so $w \notin A_p(\Omega)$. To show that $\sigma = w^{-p'/p}$ is doubling on small interior cubes we only need to note that $w \in A_p(\Omega)$ implies that $\sigma \in A_{p'}(\Omega)$.

(7) Definition. We define the maximal function associated to Ω by

$$M_\Omega f(x) = \sup_{\substack{Q, 3Q \subset \Omega \\ x \in Q}} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

The supremum is taken over all small interior cubes containing x. We note that M_Ω is zero outside of Ω and $M_\Omega f(x) \geq f(x)$ for almost every x since there is a sequence of small interior cubes converging to x.

(8) THEOREM. *Let $p > 1$. Each of the following statements imply the next.*

(i) $w \in A_p(\Omega)$.

(ii) $w \in$ *weak* $A_p(\Omega)$ *and w is doubling on small interior cubes.*

(iii) M_Ω *is weak-type (p,p) with respect to w, that is*

$$w(\{x \in \Omega : M_\Omega f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_\Omega |f(x)|^p w(x)\, dx.$$

(iv) *There exist $\alpha, \beta \in (0,1]$ such that if $E \subset Q$ where Q is a small interior cube then $|E|/|Q| < \alpha$ implies $w(E)/w(Q) < \beta$.*

(v) *A reverse Hölder inequality on small interior cubes is satisfied by w. That is, there exists an $\varepsilon > 0$ and a constant C such that*

$$\left(\frac{1}{|Q|}\int_Q w^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \leq \frac{C}{|Q|}\int_Q w$$

for all small interior cubes Q.

PROOF. (i) $\implies$ (ii). This was shown in (6).

(ii) $\implies$ (iii) Let K be a compact subset of $\{x \in \Omega : M_\Omega f(x) > \lambda\}$. We cover K with all small interior cubes such that

$$\frac{1}{|Q|}\int_Q |f(y)|\, dy > \lambda.$$

Since K is compact we may as well assume that there are a finite number of such cubes. By the Vitali covering lemma [**6**, p. 9], there exists a sequence of disjoint small interior cubes Q_1,Q_2, … Q_m such that $K \subset \bigcup_{k=1}^m 3Q_k$. Since w is doubling on small interior cubes,

$$w(K) \leq \sum_{k=1}^m w(3Q_k) \leq C \sum_{k=1}^m w(Q_k).$$

By the definition of the cubes Q_k and Hölder's inequality we have

$$|Q_k| \leq \frac{1}{\lambda}\int_{Q_k} |f(y)|\, dy \leq \frac{1}{\lambda}\left(\int_{Q_k} |f(y)|^p w(y)\, dy\right)^{1/p} \sigma(Q_k)^{1/p'}.$$

Since $w \in$ *weak* $A_p(\Omega)$, we have

$$\begin{aligned} w(K) &\leq C \sum_{k=1}^{m} \frac{w(Q_k)}{|Q_k|^p} |Q_k|^p \\ &\leq C \sum_{k=1}^{m} \left(\frac{w(Q_k)}{|Q_k|^p} \sigma(Q_k)^{p/p'} \right) \frac{1}{\lambda^p} \int_{Q_k} |f(y)|^p w(y)\, dy \\ &\leq C \sum_{k=1}^{m} \frac{1}{\lambda^p} \int_{Q_k} |f(y)|^p w(y)\, dy \\ &\leq \frac{C}{\lambda^p} \int_{\Omega} |f(y)|^p w(y)\, dy, \end{aligned}$$

because the cubes Q_k are disjoint small interior cubes of Ω. To finish the proof, we take a sequence of compact sets increasing to $\{x \in \Omega : M_\Omega f(x) > \lambda\}$.

(iii) $\Longrightarrow$ (iv) Let Q be a small interior cube of Ω. Let E be a subset of Q. Let $f = \chi_{Q-E}$ and replace λ in the weak-type inequality with $2^{-1} f_Q$. Then we get

$$\left(\frac{|Q-E|}{|Q|} \right)^p w(Q) \leq C w(Q-E).$$

If $|E|/|Q| < \alpha$ then

$$\frac{w(E)}{w(Q)} \leq 1 - \frac{(1-\alpha)^p}{C} := \beta.$$

(iv) $\Longrightarrow$ (v) Here, a usual proof [**2**, p. 397—399] holds verbatim. This proof is accomplished by a Calderón-Zygmund decomposition on a given cube Q and applying the condition analogous to (iv) to selected subcubes of Q. Since a subcube of a small interior cube is a small interior cube, we may apply (iv) to obtain (v) in exactly the same way.

When $p > 1$, the Hardy-Littlewood maximal function is bounded on $L^p(w)$ if and only if $w \in A_p(\mathbf{R}^n)$. This equivalence is central in the proof of the Rubio de Francia extrapolation theorem. We do not have such an equivalence here, but we do have the following theorem and it is sufficient to give us the extrapolation theorem (11) we seek.

(9) Theorem. *Let $p > 1$. Each of the following statements imply the next*
(i) $w \in A_p(\Omega)$.
(ii) *M_Ω is bounded on $L^p(\Omega, w)$.*
(iii) *$w \in$ weak $A_p(\Omega)$.*

Proof. (i) $\Longrightarrow$ (ii) Since $w \in A_p(\Omega)$, $\sigma = w^{-p'/p} \in A_{p'}(\Omega)$ then by (8) a Reverse Hölder Inequality holds for σ on all small interior cubes Q; as in Garcia-Cuerva and Rubio de Francia [**2**, p. 399], there exists an r where $1 < r < p$ such that

$$\left(\frac{1}{|Q|} \int_Q w^{-r'/r} \right)^{\frac{r}{r'}} \leq C \left(\frac{1}{|Q|} \int_Q w^{-p'/p} \right)^{\frac{p}{p'}}.$$

Since $w \in A_p(\Omega)$ we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)&\left(\frac{1}{|Q|}\int_Q w^{-r'/r}\right)^{\frac{r}{r'}}\\&\le C\left(\frac{1}{|Q|}\int_Q w^{-p'/p}\right)^{\frac{p}{p'}}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\\&\le C\end{aligned}$$

for all small interior cubes. Hence, $w \in$ *weak* $A_r(\Omega)$. Since $w \in A_p(\Omega)$, w is doubling by (6). Thus, by (8) M_Ω is weak-type (r, r). M_Ω is bounded on $L^\infty(\Omega, w)$ so we may apply the Marcinkiewicz Interpolation Theorem [**6**, p. 272] and conclude that M_Ω is bounded on $L^p(\Omega, w)$.

(ii) $\implies$ (iii) Given a small interior cube Q and nonnegative $f \in L^p(w, \Omega)$, $M_\Omega f(x) \ge f_Q$ for every $x \in Q$. Hence,

$$(f_Q)^p w(Q) \le C\int_\Omega f(x)^p w(x)\,dx.$$

Following Stein [**5**, p. 195] we let $\varepsilon > 0$. We modify this inequality by replacing $w(x)$ on the right hand side with $w(x) + \varepsilon$. We let $f(x) = (w(x) + \varepsilon)^{-p'/p}\chi_Q$ in the modified inequality. We rearrange the terms and let $\varepsilon \to 0$ to obtain the A_p inequality for small interior cubes. Hence, $w \in$ *weak* $A_p(\Omega)$.

One of the lingering problems of A_p weight theory is that it is not *a prioi* evident what kind of functions A_p weights are. There are several characterizations of $A_p(\mathbf{R}^n)$ weights but each characterization is equally uncheckable. Thus, necessary or sufficient conditions that are checkable are of great interest. As the following examples illustrate, one of the advantages in the definition of the classes of local weights that we have presented here is that we are able to find some sufficient conditions on functions $w \ge 0$ without much effort.

Example. Let $\gamma \in \mathbf{R}$. If $d(x) = \operatorname{dist}(x, b\Omega)$ where $b\Omega$ is the boundary of Ω then $d^\gamma \in$ *weak* $A_p(\Omega)$ for all $p \ge 1$. This is a consequence of (2)—the distance function is essentially constant on small interior cubes.

Example. Suppose we use balls instead of cubes in our definition of *weak* $A_p(\Omega)$. If $\Omega \subset \mathbf{C}$ and $w \ge 0$ is superharmonic then $w \in$ *weak* $A_1(\Omega) \subset$ *weak* $A_p(\Omega)$ for all $p \ge 1$. We can see this by considering a small interior ball $B_r \subset \Omega$ with radius r. Let $z \in B_r$. Let $B_R(z)$ be the smallest ball centered at z containing B_r. Since B_r is a small interior ball $B_R(z) \subset \Omega$. In fact, $r \le R \le 2r$. Then,

$$\frac{1}{|B_r|}\int_{B_r} w \le \frac{4}{B_R(z)}\int_{B_R(z)} w \le 4w(z)$$

by the superharmonicity of w.

Example. If $w \ge 0$ is superharmonic on $\Omega \subset \mathbf{C}$ then w is doubling on small interior balls. Let $B_R(z)$ be a small interior ball. By the superharmonicity of w

$$\int_0^{2\pi} w(z + 3re^{i\theta})\,d\theta \le \int_0^{2\pi} w(z + re^{i\theta})\,d\theta.$$

and so

$$\int_0^R \int_0^{2\pi} w(z+3re^{i\theta})r\,d\theta dr \leq \int_0^R \int_0^{2\pi} w(z+re^{i\theta})r\,d\theta dr.$$

By a change of variables we have

$$\frac{1}{9}\int_0^{3R} \int_0^{2\pi} w(z+re^{i\theta})r\,d\theta dr \leq \int_0^r \int_0^{2\pi} w(z+re^{i\theta})r\,d\theta dr.$$

Therefore

$$\int_{B_{3R}(z)} w \leq 9\int_{B_R(z)} w.$$

Example. Although nonnegative superharmonic functions are in *weak* $A_1(\Omega)$ and are doubling on small interior balls, it is not true that they are necessarily elements of $A_p(\Omega)$. Consider $\Omega = \{z : |z| < 1\} \subset \mathbf{C}$ and $w(z) = 1 - |z|^2$. By (6) for w to be an element of $A_p(\Omega)$, $w^{-p'/p}$ must be doubling on small interior balls. If $1 < p \leq 2$ then $\int_{B_{3r}(0)} w^{-p'/p}$ is unbounded as $r \to 1/3$ whereas $\int_{B_r(0)} w^{-p'/p}$ is bounded as $r \to 1/3$. Hence, $w \notin A_p(\Omega)$ for $p \leq 2$.

Finally, we present the extrapolation theorem on domains in $\mathbf{R}^n$. We note that unlike the Rubio de Francia extrapolation theorem, (11) has two classes of weights: $A_p(\Omega)$ and *weak* $A_p(\Omega)$, as a result of (9). The proof uses the following lemma.

(10) Lemma [**4**]. *Let S be a sublinear operator bounded on $L^p(\mu)$. Suppose that $Sf \geq 0$ for all nonnegative $f \in L^p(\mu)$. Then for every nonnegative $u \in L^p(\mu)$ there exists a function U such that*
(i) $u(x) \leq U(x)$ *a.e.*
(ii) $\|U\|_p \leq 2\|u\|_p$
(iii) $SU(x) \leq CU(x)$ *a.e.*

(11) Theorem. *Let $1 \leq r < \infty$. If T is a bounded operator on $L^r(\Omega, w)$ for all $w \in$ weak $A_r(\Omega)$, then T is a bounded operator on $L^p(\Omega, w)$ for all $w \in A_p(\Omega)$ and for all p, $1 < p < \infty$.*

Proof. The proof given is similar to those given in the global case [**1**] and [**4**]; see also Grafakos [**3**, p. 718]. We show how our hypotheses are used and how the conclusion is reached in the case $1 \leq r < p < \infty$. One makes the same kinds of changes to the global proof for the case $1 < p < r < \infty$.

Let $w \in A_p(\Omega)$. First, we write

$$\begin{aligned}\left(\int_\Omega |Tf|^p w\right)^{\frac{1}{p}} &= \left(\int_\Omega (|Tf|^r)^{\frac{p}{r}} w\right)^{\frac{r}{p}\frac{1}{r}} \\ &= \sup\left\{\left(\int_\Omega |Tf|^r |g| w\right)^{\frac{1}{r}} : \|g\|_{L^{(\frac{p}{r})'}(\Omega,w)} \leq 1\right\}.\end{aligned} \tag{12}$$

Taking $|g|$ for u in the lemma, we can obtain G such that $|g| \leq G$ and $\|G\|_{L^q(\Omega,w)} \leq 2\|g\|_{L^q(\Omega,w)}$ where $q = (p/r)' = p/(p-r)$, provided there exists a sublinear operator S bounded on $L^q(\Omega, w)$. If we define S by

$$Sg = \left[M_\Omega(g^t w)w^{-1}\right]^{\frac{1}{t}}$$

where $t=(p-1)/(p-r)$, then S is bounded on $L^q(w)$. We have that S is bounded on $L^q(\Omega,w)$ because $w\in A_p(\Omega)$ implies $w^{-p'/p}=\sigma\in A_{p'}(\Omega)$ which in turn implies M_Ω is bounded on $L^{p'}(\Omega,\sigma)$ by (9). By (iii) in (10), $SG(x)\le CG(x)$ for almost every x. By the definition of S, for Q any small interior cube

$$\frac{1}{|Q|}\int_Q G(x)^t w(x)\,dx \le CG(x)^t w(x) \tag{13}$$

for almost every $x\in Q$. By Hölder's inequality,

$$\frac{1}{|Q|}\int_Q Gw \le \left(\frac{1}{|Q|}\int_Q G^t w\right)^{1/t}\left(\frac{1}{|Q|}\int_Q w\right)^{(t-1)/t} \tag{14}$$

Also, by rearranging (13) to get an inequality for G^{-1}, we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (Gw)^{-r'/r}\right)^{r/r'} &\le \left(\frac{1}{|Q|}\int_Q G^t w\right)^{-1/t}\left(\frac{1}{|Q|}\int_Q w^{\frac{r'}{r}\left(\frac{1}{t}-1\right)}\right)^{r/r'}\\ &= \left(\frac{1}{|Q|}\int_Q G^t w\right)^{-1/t}\left(\frac{1}{|Q|}\int_Q w^{-p'/p}\right)^{\frac{p}{p'}\left(\frac{t-1}{t}\right)}\end{aligned}$$

We multiply the inequalities (13) and (14) and find that $Gw\in$ *weak* $A_r(\Omega)$ with A_r constant majorized by a fixed power of the A_p constant of w. Thus, for $\|g\|_{L^{(p/r)'}(\Omega,w)}\le 1$ we have

$$\begin{aligned}\left(\int_\Omega |Tf|^r|g|w\right)^{\frac{1}{r}} &\le \left(\int_\Omega |Tf|^r Gw\right)^{\frac{1}{r}}\\ &\le C\left(\int_\Omega |f|^r Gw\right)^{\frac{1}{r}}\\ &\le C\left(\int_\Omega (|f|^r)^{\frac{p}{r}} w\right)^{\frac{1}{p}}\left(\int_\Omega G^{\left(\frac{p}{r}\right)'} w\right)^{\frac{1}{r\left(\frac{p}{r}\right)'}}\\ &\le C\|f\|_{L^p(\Omega,w)}\|G\|^{1/r}_{L^{(p/r)'}(\Omega,w)}\\ &\le C\|f\|_{L^p(\Omega,w)}\|g\|^{1/r}_{L^{(p/r)'}(\Omega,w)}\\ &\le C\|f\|_{L^p(\Omega,w)}.\end{aligned}$$

Putting this inequality together with (12) gives us that T is bounded from $L^p(\Omega,w)$ to $L^p(\Omega,w)$.

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