Anomalous Dimensions of Anisotropic Gauge Theory Operators

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The anomalous dimensions of the anisotropic dimension four operators in SU(N) gauge theory coupled to fermions are calculated to lowest order. The possibility of comparison with numerical simulations is pointed out.

1. Introduction

Although great effort has been put into the numerical simulation of discretized gauge theories, there has been a notable paucity of quantitative analytic results with which to compare the simulations. The running of the coupling constant according to the renormalization group coefficient β, and some clever generalizations involving anisotropic cutoffs [1], are the only examples that come to mind.

We have calculated the anomalous dimensions of certain gauge-invariant, generally anisotropic, dimension four operators in SU(N) gauge theories coupled to fermions, using techniques of continuum perturbation theory. We shall argue that these anomalous dimensions ought also to be readily calculable from the simulations. If such calculations were done and agreed with the predictions, they would supply highly nontrivial verification that renormalized quantum field theory at weak coupling does indeed describe the short-distance behavior of the (nonperturbative) lattice theory. There has been some doubt expressed about this recently [2]. Even if one believes that such checks are redundant in principle, their success would be welcome reassurance that in practice the lattice size has been taken sufficiently small to reproduce the continuum.

We shall first give an account of the technique of calculation and of the results obtained, and then sketch how a comparison with numerical lattice gauge theory results could be implemented.

2. The anisotropic anomalous dimensions

Let us first consider the pure gluon theory, specified by the lagrangian

$$\mathscr{L} = -\tfrac{1}{4} \sum_{a,\mu,\nu} F^a_{\mu\nu} F^a_{\mu\nu} + (\text{gauge fixing}) + (\text{ghosts}) , \tag{2.1}$$

with the gluon field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu . \tag{2.2}$$

(We work in euclidean spacetime throughout.) A basis for the gauge-invariant dimension four anisotropic operators we wish to consider is provided by the six objects

$$O_{12} \equiv \tfrac{1}{2} \sum_a F^a_{12} F^a_{12} , \quad O_{13} \equiv \tfrac{1}{2} \sum_a F^a_{13} F^a_{13} , \dots . \tag{2.3}$$

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By reflection symmetry there is no mixing between these operators and e.g., $\sum_a F^a_{12}F^a_{13}$, so that this set of operators is closed under renormalization [#1].

Now, upon renormalization there are evidently three independent operator mixings which can occur:

$$\mu\frac{\partial}{\partial\mu}O_{12}=-\gamma_a O_{12}-\gamma_b(O_{13}+O_{14}+O_{23}+O_{24})-\gamma_c O_{34}\,. \tag{2.4}$$

It is worth noting, however, that there are certain identifiable linear combinations of the $O_{\mu\nu}$ multiplicatively renormalized, with anomalous dimensions that are combinations of γ_a, γ_b and γ_c. These are

$$\begin{aligned}G_{\rm I}&\equiv O_{12}+O_{13}+O_{14}+O_{23}+O_{24}+O_{34}\,,\\ G_{\rm II}&\equiv O_{12}-O_{34}\,,\\ G_{\rm III}&\equiv O_{12}+O_{34}-\tfrac{1}{2}(O_{13}+O_{14}+O_{23}+O_{24})\,,\end{aligned} \tag{2.5}$$

with

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}G_{\rm I}&=-(\gamma_a+4\gamma_b+\gamma_c)G_{\rm I}\,,\\ \mu\frac{\partial}{\partial\mu}G_{\rm II}&=-(\gamma_a-\gamma_c)G_{\rm II}\,,\\ \mu\frac{\partial}{\partial\mu}G_{\rm III}&=-(\gamma_a-2\gamma_b+\gamma_c)G_{\rm III}\,.\end{aligned} \tag{2.6}$$

There are two other operators of the same general form as $G_{\rm II}$, and one of the same form as $G_{\rm III}$, which we do not display. Now, $G_{\rm I}$ is just F^2, the lagrangian. Its anomalous dimension is closely related to the conventional beta function [3]. The three $G_{\rm II}$-type operators are linear combinations of the diagonal elements of the energy–momentum tensor, and are therefore expected on general grounds to have vanishing anomalous dimensions.

The actual computation of the anomalous dimensions is straightforward, though somewhat laborious; it involves evaluating the diagrams shown in fig. 1. Before we present our results, however, a few remarks about operator mixing in gauge theories are in order.

As is well known, the renormalization of a composite operator in general requires counterterms proportional to other composite operators of the same (or lower) mass dimension and carrying the same quantum numbers. Obviously it is very helpful if the number of operators which mix together is fairly small, so that the calculation of the matrix of renormalization constants is not too difficult. Life would be most pleasant in this regard if the counterterms needed for a gauge-invariant operator (i.e., one formally invariant under the usual gauge transformations of the elementary fields) were themselves gauge-invariant. Indeed it is somewhat frightening to contemplate the alternative, since the number of gauge-noninvariant operators which *a priori* might mix with a given gauge-invariant one is usually enormous.

Actually, the truth is somewhere in between. We now briefly summarize the result of a rather involved analysis, which is fully presented in ref. [4]. Define type I operators to be the formally gauge-invariant operators of interest. It can be shown that operators which mix with type I operators (and are not themselves type I) are of two special types: type IIa operators, which are BRS variations of some other operators, and type IIb operators, which formally vanish by the equations of motion. These three sets of operators together form a closed set under renormaliza-

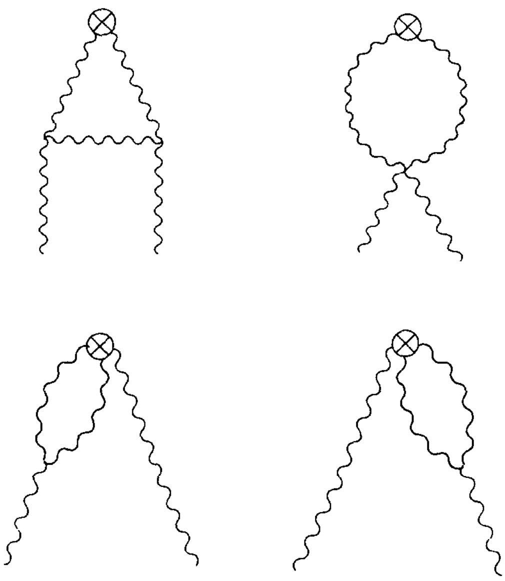

Fig. 1.

[#1] Other closed sets of anisotropic operators, containing $F_{12}F_{13}$, etc., could also be considered.

tion. In this basis of operators the renormalization matrix Z_{ab}, defined by

$$O_a^{\text{bare}} = Z_{ab} O_b^{\text{ren}} \tag{2.7}$$

takes a simple form. The operators of type II mix only with each other. Thus Z_{ab} has the block triangular form

$$Z_{ab} = \begin{pmatrix} Z_{\text{I,I}} & Z_{\text{I,II}} \\ 0 & Z_{\text{II,II}} \end{pmatrix}. \tag{2.8}$$

So if we are willing to work on-shell and to refrain from looking at BRS null states, the renormalization constants $Z_{\text{I,I}}$ that connect type I operators among themselves are adequate. They yield the physically correct, gauge-independent anomalous dimensions for the type I operators.

The physical on-shell matrix elements, i.e. those with all external (momenta)^2 set to zero and with physical polarization vectors attached, of all type II operators vanish. Thus we can isolate the type I counterterms $Z_{\text{I,I}}$ simply by ignoring the gauge-non-invariant operators, and evaluating all matrix elements on-shell. Finally, from $Z_{\text{I,I}}$ we compute the physically relevant gauge-independent anomalous dimensions. This is an eminently practical scheme, and is the one we employed.

The calculation is slightly complicated by the fact that we must evaluate all matrix elements on-shell. Indeed, the two-gluon matrix elements of $O_{\mu\nu}$ at zero momentum transfer vanish on-shell, so to see this operator we must either consider three-gluon matrix elements or evaluate the operator insertions at non-zero momentum transfer. We chose the latter alternative.

Our results are, for pure SU(N) gauge theory

$$\gamma_{\text{a}} = \frac{g^2 C_{\text{A}}}{6(8\pi^2)}, \quad \gamma_{\text{b}} = -\frac{g^2 C_{\text{A}}}{8\pi^2}, \quad \gamma_{\text{c}} = \frac{g^2 C_{\text{A}}}{6(8\pi^2)}, \tag{2.9}$$

where C_{A} is the Casimir invariant of the adjoint representation of SU(N):

$$C_{\text{A}} \delta^{ab} = f^{acd} f^{bcd}, \tag{2.10}$$

with the f^{abc} the structure constants of the group. These results are consistent with the general expectations discussed above, namely $\gamma_{\text{a}} = \gamma_{\text{c}}$ and $\gamma_{\text{a}} + 4\gamma_{\text{b}} + \gamma_{\text{c}} = 2\beta(g)/g$. We also checked the results by calculating the γ's in the general Fermi-type gauge, and finding that they are independent of the gauge parameter.

To include massless fermions, we augment the lagrangian (2.1) with the term

$$\mathscr{L}_{\text{f}} = \text{i} \sum_{\mu} \bar{\psi} \gamma_\mu \text{D}_\mu \psi, \tag{2.11}$$

where

$$\text{D}_\mu = \partial_\mu - \text{i} g A_\mu^a t^a \tag{2.12}$$

and the t^a are generators of SU(N) in the fermion representation. The four new operators we must include are

$$P_1 \equiv \text{i}\bar{\psi}\gamma_1 \text{D}_1 \gamma, \quad P_2 \equiv \text{i}\bar{\psi}\gamma_2 \text{D}_2 \psi, \quad \ldots . \tag{2.13}$$

We define additional mixing coefficients by

$$\mu \frac{\partial}{\partial \mu} P_1 = -\gamma_{\text{d}} P_1 - \gamma_{\text{e}}(P_2 - P_3 + P_4)$$
$$- \gamma_{\text{f}}(O_{12} + O_{13} + O_{14}) - \gamma_{\text{h}}(O_{23} + O_{24} + O_{34}), \tag{2.14}$$

$$\mu \frac{\partial}{\partial \mu} O_{12} = (\text{old terms}) - \gamma_{\text{k}}(P_1 + P_2) - \gamma_{\ell}(P_3 + P_4).$$

Evaluating the additional graphs of fig. 2 then yields

$$\gamma_{\text{a}} = \frac{g^2}{8\pi^2} \left(\tfrac{1}{6} C_{\text{A}} + \tfrac{4}{3} T_{\text{R}}\right),$$

$$\gamma_{\text{b}} = -\frac{g^2 C_{\text{A}}}{8\pi^2}, \quad \gamma_{\text{c}} = \frac{g^2 C_{\text{A}}}{6(8\pi^2)},$$

$$\gamma_{\text{d}} = \frac{2g^2 C_{\text{R}}}{8\pi^2}, \quad \gamma_{\text{e}} = -\frac{2g^2 C_{\text{R}}}{3(8\pi^2)},$$

$$\gamma_{\text{f}} = -\gamma_{\text{h}} = \frac{4g^2 T_{\text{R}}}{3(8\pi^2)}, \quad \gamma_{\text{k}} = -\gamma_{\ell} = \frac{g^2 C_{\text{R}}}{3(8\pi^2)}, \tag{2.15}$$

where T_{R} and C_{R} are the index and Casimir invariant, respectively, of the fermion representation:

$$T_{\text{R}} \delta^{ab} = \text{tr}(t^a t^b), \quad C_{\text{R}} \delta_{ij} = (t^a t^a)_{ij}. \tag{2.16}$$

Again, we have a sum rule relating our anomalous dimensions to the ordinary β function: $\gamma_{\text{a}} + 4\gamma_{\text{b}} + \gamma_{\text{c}} = 2\beta(g)/g$. Also, there must be four operators with zero anomalous dimension, corresponding to the diagonal components of the energy–momentum tensor. This last condition provides a highly nontrivial check on the calculation, as it corresponds to the vanishing

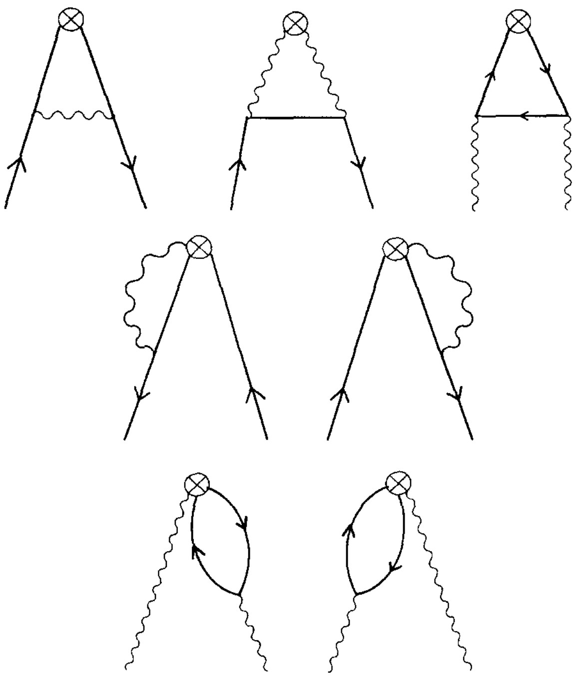

Fig. 2.

of the determinant of the 9×9 matrix of anomalous dimensions.

As another check on the calculation we also computed the insertions of the P_μ into the Green function with two external fermions and one gluon. This amounts to considering the pieces of the counterterms proportional to $\bar\psi\gamma_\mu A_\mu \psi$ rather than $\bar\psi\gamma_\mu\,\partial_\mu\psi$. The coefficients of these must come out the same, in order that they may be assembled into a gauge-invariant counterterm for $\bar\psi\gamma_\mu\,\mathrm{D}_\mu\psi$.

3. Applications

The operators $O_{\mu\nu}$ appear prominently in lattice gauge theory, as the limit, when the lattice spacing is small, of the operators

$$\frac{1}{a^2g^2}\,\mathrm{tr}\,(U_{\mu\nu}-1)(U^\dagger_{\mu\nu}-1)\sim O_{\mu\nu}\,. \qquad (3.1)$$

Here $U_{\mu\nu}$ is the Wilson line integral around a plaquette in the $\mu\nu$ plane, and a^2 is the area enclosed by such a plaquette. Similarly, the fermion operators we have calculated above are simple objects on the lattice.

The correlation function between $Q_{\mu\nu}$ at nearby points, and possibly with different indices, can be calculated using the operator product expansion. The leading singularity as $x\to0$ is of the form

$$\langle G_\mathrm{A}(x)G_\mathrm{B}(0)\rangle \approx \mathscr{C}_\mathrm{AB}(x,g;\mu)\mathbf{1}+\ldots \qquad (3.2)$$

for any of the multiplicatively renormalized combinations G_A appearing in (2.5). Here the dots represent higher-dimension operators, and $\mathscr{C}_\mathrm{AB}$ is a c-number function obeying the renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-\gamma_\mathrm{A}-\gamma_\mathrm{B}\right)\mathscr{C}_\mathrm{AB}(x,g;\mu)=0\,. \qquad (3.3)$$

Eq. (3.3) may be solved to yield

$$\mathscr{C}_\mathrm{AB}(\lambda^{-1}x,g)=\lambda^8\left(\frac{\bar g(t)}{g}\right)^{(\gamma^0_\mathrm{A}+\gamma^0_\mathrm{B})/2b_0}\mathscr{C}_\mathrm{AB}(x,\bar g(t)) \qquad (3.4)$$

to lowest nontrivial order, where $\gamma_\mathrm{A}=g^2\gamma^0_\mathrm{A}+\ldots$, $\beta=-b_0g^3+\ldots$, $t\equiv\ln\lambda$, and $\bar g(t)$ is the running coupling as usually defined.

For sufficiently short distances, such that the effective coupling becomes small, we may calculate $\mathscr{C}_\mathrm{AB}$ approximately using perturbation theory. The correlation functions of the operators appearing in (2.5) may be obtained from the perturbative results for the $O_{\mu\nu}$. In lowest order perturbation theory the three distinct cases are, e.g.,

$$\langle O_{12}(x)O_{12}(0)\rangle=\frac{(x_1^2+x_2^2-x_3^2-x_4^2)^2}{2\pi^4(x^2)^6}\,,$$

$$\langle O_{12}(x)O_{13}(0)\rangle=\frac{2x_2^2x_3^2}{\pi^4(x^2)^6}\,,$$

$$\langle O_{12}(x)O_{34}(0)\rangle=0\,, \qquad (3.5)$$

where we have neglected δ-function singularities. Combining eqs. (2.6), (3.4), and (3.5) we obtain predictions for the correlation functions of interest.

Similarly, we find for the fermion operator products in lowest order perturbation theory

$$\langle P_\mu(x)P_\nu(0)\rangle=\frac{16}{\pi^4(x^2)^6}\,[2x_\mu^2x_\nu^2 -x^2x_\mu x_\nu g_{\mu\nu}-\tfrac14x^2(x_\mu^2+x_\nu^2)+\tfrac1{16}(x^2)^2]\,; \qquad (3.6)$$

no summation is implied in this expression.

The anisotropic operators we have been discussing are in principle readily accessible on the lattice, as we have stressed before. However, one subtlety ought to be mentioned. The correlation functions (3.5) and (3.6) vary rapidly with distance (as $1/x^8$) even in free field theory. The interesting QCD modifications of this behavior are minor by comparison; only powers of logarithms at small x. Since there is some ambiguity concerning the correct continuum interval x to associate with pairs of extended operators such as our plaquette operators on the lattice, an accurate comparison between the analytic predictions and the numerical results for the change of the correlation function with distance would seem to be quite difficult. Fortunately, there is an alternative procedure: to compare the correlation functions *in a fixed geometry, as a function of coupling*. The variation in this case will come only from the anomalous dimensions of interest, which thereby become practically accessible.

Comparison between the analytically calculated, continuum correlation functions and lattice simulations clearly is called for.

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