6-27-1996

Light Front QCD in (1+1)-Dimensions Coupled to Chiral Adjoint Fermions

David G. Robertson
Otterbein University

Stephen S. Pinksy

ELSEVIER

27 June 1996

Physics Letters B 379 (1996) 169–178

PHYSICS LETTERS B

Light-front QCD_{1+1} coupled to chiral adjoint fermions

Stephen S. Pinsky, David G. Robertson
Department of Physics, The Ohio State University, Columbus, OH 43210, USA

Received 27 December 1995; revised manuscript received 16 February 1996
Editor: M. Dine

Abstract

We consider $SU(N)$ gauge theory in 1+1 dimensions coupled to chiral fermions in the adjoint representation of the gauge group. With all fields in the adjoint representation the gauge group is actually $SU(N)/Z_N$, which possesses nontrivial topology. In particular, there are N distinct topological sectors and the physical vacuum state has a structure analogous to a θ vacuum. We show how this feature is realized in light-front quantization for the case $N = 2$, using discretization as an infrared regulator. In the discretized form of the theory the nontrivial vacuum structure is associated with the zero momentum mode of the gauge field A^+. We find exact expressions for the degenerate vacuum states and the analog of the θ vacuum. The model also possesses a condensate which we calculate. We discuss the difference between this chiral light-front theory and the theories that have previously been considered in the equal-time approach.

1. Introduction

The unique features of light-front quantization [1] make it a potentially powerful tool for the study of QCD. Of primary importance in this approach is the apparent simplicity of the vacuum state. Indeed, naive kinematical arguments suggest that the physical vacuum is trivial on the light-front. This cannot really be true, of course, particularly in view of the important physics associated with the QCD vacuum. Thus it is crucial to understand the ways in which vacuum structure can be manifested in light-front quantization.

There has recently been significant progress in this regard. If one uses discretization [2] as an infrared regulator (i.e. imposes periodic or antiperiodic boundary conditions on some finite interval in x^-), then any vacuum structure must necessarily be connected with the $k^+ = 0$ Fourier modes of the fields. Studies of model field theories have shown that the zero modes can in fact support certain kinds of vacuum structure; the long range phenomena of spontaneous symmetry breaking [3] as well as the topological structure [4,5] can in fact be reproduced with a careful treatment of the zero mode(s) of the fields in a quantum field theory defined in a finite spatial volume and quantized at equal light-front time.

These phenomena are realized in quite different ways. For example, spontaneous breaking of Z_2 symmetry in ϕ^4_{1+1} occurs via a *constrained* zero mode of the scalar field [6]. There the zero mode satisfies a nonlinear constraint equation that relates it to the dynamical modes in the problem. At the critical coupling a bifurcation of the solution occurs. These solutions in turn lead to new operators in the Hamiltonian which break the Z_2 symmetry at and beyond the critical coupling. Quite separately, a *dynamical* zero mode was shown in Ref. [4] to arise in pure $SU(2)$ Yang-Mills theory in 1+1 dimensions. A complete fixing of the gauge leaves the theory with one degree of freedom, the zero mode of the vector potential A^+. The theory

PII S0370-2693(96)00252-3

has a discrete spectrum of zero-P^+ states corresponding to modes of the flux loop around the finite space. Only one state has a zero eigenvalue of the energy P^-, and is the true ground state of the theory. The nonzero eigenvalues are proportional to the length of the spatial box, consistent with the flux loop picture. This is a direct result of the topology of the space. As the theory considered there was a purely topological field theory, the exact solution was identical to that in the conventional equal-time approach on the analogous spatial topology [7].

In the present work we shall focus on the vacuum structure and condensate of QCD_{1+1} coupled to adjoint fermionic matter. For vector-like coupling this model has been studied in the limit of large N_c [8], for $N_c = 2$ at finite temperature [9], and for 2 and 3 colors in equal-time quantization in the small-volume limit [10]. It is interesting in that it possesses a vacuum structure analogous to a θ vacuum. As first shown in Ref. [11], for $SU(N)$ gauge fields the vacuum has a Z_N topological structure. Furthermore, for $N = 2$ there is a nonvanishing bilinear condensate [9].

We shall consider here a chiral version of the model described in [10]. In the conventional light-front approach [1], the chiral nature of the theory is automatic for massless fermions in 1+1 dimensions [12]. In order to obtain the theory discussed in Ref. [10], it is necessary to include additional degrees of freedom, initialized along a second null plane, which represent the left-handed particles. We hope to report on this in the near future. The topological classification of the vacua is unaffected by the chiral nature of the theory, however. Thus we expect to find N degenerate vacua for $SU(N)$ gauge fields even in the chiral model. For the case $N = 2$ considered here we shall indeed find two vacuum states. As suggested above, the physics of these states is closely connected to the only zero mode in the theory, that of A^+. The properties of this mode, in turn, are tied up with issues of gauge fixing, Gribov horizons, etc.

It is always a delicate matter to consistently formulate a chiral gauge theory with an anomaly. There is an extensive literature on this subject, including recent results for QCD_{1+1} [13]. We will not dwell on this issue since it does not appear to be central to the structure of the condensate.

[1] That is, with all fields initialized on a single null plane.

The chiral nature of the theory we consider implies that any condensate that we will find will be fundamentally different in structure from the one found in [10]. As discussed in [10], by considering the spectral flow of the fermions under large gauge transformations it can be shown that the vector-like model has a $\bar{\Psi}\Psi$ condensate for $SU(2)$ and a $(\bar{\Psi}\Psi)^2$ condensate for $SU(3)$. As we shall see, an analogous argument leads us to anticipate a condensate for Ψ alone in the chiral model. Since we are able to find the exact vacua in the light-front formulation we are able to find an exact expression for the condensate.

Similar theories coupled to adjoint scalars have also been studied recently [14]. Here the scalar field can be thought of as the $k_\perp = 0$ remnant of the transverse gluon component in QCD_{2+1}. The study of these theories is part of a long-term program to attack QCD_{3+1} through the zero mode sectors starting with studies of lower dimensional theories which are themselves zero mode sectors of higher dimensional theories. A complete gauge fixing has recently been given for QED_{3+1} which further supports this program [15]. In all of these cases, the central problem was to disentangle the dependent from the independent fields in the context of a particular gauge fixing.

These issues become phenomenologically interesting in the context of recent work on collinear QCD, or the "tube" model [16]. In this approach one considers dimensionally reduced QCD, which takes the form of an effective two-dimensional theory. While these theories contain fundamental fermions, the connection between the spectra of QCD_{1+1} with fundamental and adjoint fermions [17] makes them an interesting subject. It would also be natural to consider dimensionally reduced supersymmetric theories, which would include adjoint fermions of the type discussed here. In addition, dimensionally reduced pure glue QCD already has an adjoint scalar which might behave similarly to adjoint fermions [14]. The issue of the spectral density of states in theories of this type arises in contexts such as matrix models. Finally, this entire class of models is very interesting for studying issues of confinement, screening and the comparison of the behavior of massive and massless theories with fractional charges [18].

The remainder of the paper is organized as follows. In the next section we define the theory, including the gauge fixing, and outline our calculational scheme.

Section 3 is devoted to the definition of the currents and charge operators, which are important for defining a suitable physical subspace. Next we study the vacuum sector of the theory and find the two ground states. In Section 5 we discuss the condensate and obtain an exact expression for it. Section 6 contains some discussion and directions for future work.

2. Definition of the theory and gauge fixing

We consider an $SU(N)$ gauge field coupled to adjoint fermions in one space and one time dimension. Since all fields transform according to the adjoint representation, gauge transformations that differ by an element of the center of the group actually represent the same transformation and so should be identified. Thus the gauge group of the theory is $SU(N)/Z_N$, which has nontrivial topology: $\Pi_1[SU(N)/Z_N] = Z_N$, so that one expects N topological sectors. This situation differs from the case when the matter fields are in the fundamental representation, where the gauge group is $SU(N)$ and the first homotopy group is trivial.

The Lagrangian for the theory is

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + \tfrac{1}{2}\mathrm{Tr}(\bar{\psi}\gamma^{\mu} \overleftrightarrow{D}_{\mu}\, \psi)\,, \tag{2.1}$$

where $D_\mu = \partial_\mu + ig[A_\mu,\ ]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. We employ light-front quantization, defining $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ and taking x^+ to be the evolution parameter. A convenient representation of the gamma matrices is $\gamma^0 = \sigma^2$ and $\gamma^1 = i\sigma^1$, where σ^a are the Pauli matrices. With this choice, the Fermi field may be taken to be hermitian. It is natural in light-front quantization to break the Fermi field into two components

$$\Psi_\pm \equiv \tfrac{1}{2}\gamma^0\gamma^\pm\Psi\,; \tag{2.2}$$

in two dimensions (only) these are the same as chiral projections, so that

$$\tfrac{1}{2}\gamma^0\gamma^+\Psi = \begin{pmatrix}\Psi_R\\ 0\end{pmatrix}, \quad \tfrac{1}{2}\gamma^0\gamma^-\Psi = \begin{pmatrix}0\\ \Psi_L\end{pmatrix}. \tag{2.3}$$

When the fermions are massive, Ψ_+ appears to be the only independent degree of freedom. For the massless theory considered here, both Ψ_+ and Ψ_- must be considered to be independent fields [12].

We shall focus on the case $N = 2$, in which case the matrix representation of the fields makes use of the $SU(2)$ generators $\tau^a = \sigma^a/2$. It is convenient to introduce a color helicity basis, defined by $\tau^\pm \equiv \tau^1 \pm i\tau^2$ with τ^3 unchanged. These satisfy $[\tau^+, \tau^-] = \tau^3$ and $[\tau^3, \tau^\pm] = \pm\tau^\pm$. In terms of this basis the matrix-valued fields are given by, for example,

$$A^\mu = A_3^\mu\tau^3 + A_+^\mu\tau^+ + A_-^\mu\tau^-\,, \tag{2.4}$$

where $A_\pm^\mu \equiv A_1^\mu \pm iA_2^\mu$. (Note that $(A_+^\mu)^\dagger = A_-^\mu$.) The Fermi field will be similarly written as

$$\Psi_{R/L} = \psi_{R/L}\tau^3 + \phi_{R/L}\tau^+ + \phi_{R/L}^\dagger\tau^-\,, \tag{2.5}$$

with $\phi_{R/L} \equiv \Psi_{R/L}^1 + i\Psi_{R/L}^2$. Under a gauge transformation the gauge field transforms in the usual way and the Fermi field transforms according to

$$\Psi_{R/L} \Rightarrow U\Psi_{R/L}U^{-1}\,, \tag{2.6}$$

where U is a spacetime-dependent element of $SU(2)$.

We shall regulate the theory by putting it in a light-front spatial box, $-L < x^- < L$, and imposing periodic boundary conditions for the gluon fields A^μ and anti-periodic boundary conditions for the Fermi field. In this approach, the subtle aspects of formulating the model have to do with the zero-momentum modes of the fields. It is here, also, that any nontrivial vacuum structure must reside.

In the present model the subtlety is in fixing the gauge. It is most convenient in light-front field theory to choose the light-cone gauge $A^+ = 0$. Here, however, since the gauge transformation must be periodic up to an element of the center of the gauge group (here Z_2), we cannot gauge the zero mode of A^+ to zero [19]. Thus we choose $\partial_- A^+ = 0$. We can make a further global (i.e. x^--independent) rotation so that the zero mode of A^+ has only a color 3 component,

$$A^+ = v(x^+)\tau^3 \equiv V(x^+)\,, \tag{2.7}$$

and simultaneously rotate A^- so that it has no color 3 zero mode [4].

At this stage the only remaining gauge freedom involves certain "large" gauge transformations, which we shall denote T_n. This freedom is best studied in terms of the dimensionless variable $z = gvL/\pi$, which T_n shifts by an integer:

$$T_n z T_n^{-1} = z + n\,. \tag{2.8}$$

In addition, T_n generates a space-dependent phase rotation on the matter field $\phi_{R/L}$

$$T_n \phi_{R/L} T_n^{-1} = e^{-in\pi x^-/L} \phi_{R/L}\,, \tag{2.9}$$

which however preserves the anti-periodic boundary condition on $\phi_{R/L}$. This gauge freedom is an example of the Gribov ambiguity [20]. We can use it to bring z to a finite domain, for example $0 < z < 1$ or $-1 < z < 0$. Once this is done all gauge freedom has been exhausted and the gauge fixing is completed. Only physical degrees of freedom remain.

After gauge fixing T_n is no longer a symmetry of the theory, but there is a symmetry of the gauge-fixed theory that is conveniently studied by combining T_1 with the so-called Weyl symmetry, denoted by R. Under R,

$$RzR^{-1} = -z \quad \text{and} \quad R\phi_{R/L}R^{-1} = \phi^\dagger_{R/L}\,. \tag{2.10}$$

This is also not a symmetry of the gauge-fixed theory, as it takes z out of the fundamental domain. The symmetry $T_1 R$, however, which is closely related to charge conjugation, plays an important role in the gauge-fixed theory as will be discussed in detail below.

The Hamiltonian P^- takes a very standard form

$$P^- = g \int_{-L}^{L} dx^- \mathrm{Tr}\left(AJ^+\right) + 2L\partial_+ v \partial_+ v\,, \tag{2.11}$$

where $A \equiv A^-$ and $J^+ = 1/\sqrt{2}[\Psi_R, \Psi_R]$. The field A is nondynamical and is obtained by solving Gauss' law,

$$-D_-^2 A = gJ^+\,. \tag{2.12}$$

Resolving this into its color components we have

$$-\partial_-^2 A_3 = gJ_3^+ \tag{2.13}$$

$$-(\partial_- + igv)^2 A_+ = gJ_+^+ \tag{2.14}$$

$$-(\partial_- - igv)^2 A_- = gJ_-^+\,. \tag{2.15}$$

The first of these can be solved for the normal mode part of A_3 (recall that the zero mode has been gauged away). Because of the boundary conditions and the restriction of v to a finite domain, the covariant derivatives appearing in the second and third equations have no zero eigenvalues. Thus they can be inverted to solve for A_+ and A_-. The only part of Gauss' law that remains to be implemented is the zero mode of the first equation, which reduces to the vanishing of the zero mode of J_3^+. This condition must be imposed on the states and defines the physical subspace of the theory:

$$Q_3|\mathrm{phys}\rangle = 0\,, \tag{2.16}$$

where

$$Q_3 = \int_{-L}^{L} dx^- J_3^+\,. \tag{2.17}$$

After implementing the solution of Gauss' law we have

$$P^- = \frac{g^2}{2(2\pi)^2}\pi_z^2 - g^2 \int_{-L}^{L} dx^- \mathrm{Tr}\left(J^+ \frac{1}{D_-^2} J^+\right)\,, \tag{2.18}$$

where π_z is the momentum conjugate to the quantum mechanical degree of freedom $z = gvL/\pi$, defined so that $[z, \pi_z] = i$. In this form it is clear that the dynamical variables are Ψ_R and z. We shall use a Fock space representation for the Fermi degrees of freedom and a Schrödinger representation for the z degree of freedom. Thus states will be written as tensor products of the general form $\psi(z) \otimes |\mathrm{Fock}\rangle$, and π_z will be represented as a derivative operator: $\pi_z = -i\partial_z$. The Fourier expansion of Ψ_R has the usual form

$$\psi_R = \frac{1}{2^{1/4}\sqrt{2L}} \sum_n \left(a_n e^{-ik_n^+ x^-} + a_n^\dagger e^{ik_n^+ x^-}\right) \tag{2.19}$$

$$\phi_R = \frac{1}{2^{1/4}\sqrt{2L}} \sum_n \left(b_n e^{-ik_n^+ x^-} + d_n^\dagger e^{ik_n^+ x^-}\right)\,, \tag{2.20}$$

where the sums are over the positive half odd integers and $k_n^+ = n\pi/L$. The Fock operators obey the standard commutation relations

$$\{a_n^\dagger, a_m\} = \{b_n^\dagger, b_m\} = \{d_n^\dagger, d_m\} = \delta_{n,m}\,. \tag{2.21}$$

These result in the Heisenberg equation correctly reproducing the equation of motion for Ψ_R,

$$D_+\Psi_R = \partial_+\Psi_R + ig[A, \Psi_R] = 0\,. \tag{2.22}$$

In addition, of course, $[z, \Psi_R] = [\pi_z, \Psi_R] = 0$.

3. Current operators and the physical subspace

Next let us discuss the definition of the current J^+ and the associated charge operator in more detail. The relation

$$J^+ = \frac{1}{\sqrt{2}}[\Psi_R, \Psi_R] \tag{3.1}$$

is ill-defined as it stands. We shall regulate it using a gauge invariant point splitting:

$$J^+ = \lim_{\epsilon\to 0}\frac{1}{\sqrt{2}}\left[\exp\left(-ig\int_{x^- -\epsilon}^{x^-} v\tau^3 dx^-\right)\Psi_R(x^- - \epsilon) \times \exp\left(ig\int_{x^- -\epsilon^-}^{x^-} v\tau^3 dx^-\right), \Psi_R(x^-)\right]. \tag{3.2}$$

We find that the current J^+ acquires a gauge correction

$$J^+ = J'^+ + \frac{g}{2\pi}v(x^+)\tau^3, \tag{3.3}$$

where J'^+ is the naive normal-ordered current. This result is potentially upsetting, as the charge calculated from this current would seem to have an unwanted time dependence. Note, however, that any x^--independent piece of the current J_3^+ couples to the zero mode of A_3, which has been gauged to zero [see Eq. (2.11)]. Therefore what enters the dynamics is not the full current but the current with the "anomaly" (and any other color 3 zero mode) removed. It follows that J'^+ is what will appear in the equations of motion. In particular, Gauss' law takes the form

$$-D_-^2 A = gJ'^+. \tag{3.4}$$

Since the zero mode of J_3^+ does not appear in the dynamics of the theory, one can ask how it is to be defined. As we have seen, it is necessary to define the current in such a way that its zero mode has no gauge correction. The presence of such a term would be quite unpleasant as the charge, which is supposed to project out physical states, would be time-dependent. Another property that the charge should possess is $T_1 R$ symmetry. In order to discuss this it is helpful to consider the transformation properties of the fields and the naive charge

$$Q_3' = \sum_n (d_n^\dagger d_n - b_n^\dagger b_n) \tag{3.5}$$

under T_1 and R. From Eq. (2.9) we see that T_1 gives rise to a spectral flow,

$$T_1 b_n T_1^{-1} = b_{n-1}, \quad n > 1/2 \tag{3.6}$$

$$T_1 d_n T_1^{-1} = d_{n+1} \tag{3.7}$$

$$T_1 b_{1/2} T_1^{-1} = d_{1/2}^\dagger. \tag{3.8}$$

This leads to

$$T_1 Q_3' T_1^{-1} = Q_3' - 1. \tag{3.9}$$

In addition, T_1 shifts z by unity [Eq. (2.8)]. Under R symmetry, meanwhile, we find from Eq. (2.10) that

$$R b_n R^{-1} = d_n, \tag{3.10}$$

which gives

$$R Q_3' R^{-1} = -Q_3'. \tag{3.11}$$

Its action on z is to take $z \Rightarrow -z$ [Eq. (2.10)]. Putting these together we find

$$T_1 R Q_3' R^{-1} T_1^{-1} = 1 - Q_3' \tag{3.12}$$

and

$$T_1 R z R^{-1} T_1^{-1} = -z - 1. \tag{3.13}$$

This represents a symmetry of the theory since it maps the fundamental domain $-1 < z < 0$ onto itself. In fact, $T_1 R$ represents a reflection of the fundamental domain about its midpoint $z = -1/2$, coupled with a spectral flow of the fermionic degrees of freedom. It is straightforward to check that the Hamiltonian Eq. (2.18) commutes with $T_1 R$.

Now the charge operator we use to select the physical subspace must also be invariant under $T_1 R$, so that the physical subspace is mapped into itself under the transformation. Clearly, Q_3' is not invariant and so cannot be used for this purpose. Note, however, that *two* applications of the transformation $T_1 R$ leave Q_3', as well as the fundamental domain, invariant. Thus if we define the physical subspace to consist of all states annihilated either by Q_3' or by $1 - Q_3'$, then it will be invariant under the $T_1 R$ transformation and this will represent a true symmetry of the theory. As this has all the properties we require, we shall adopt it as the definition of the physical subspace. Note that it is stable under time evolution, since $[Q_3', P^-] = 0$.

4. Vacuum states of the theory

The Fock state containing no particles will be called $|V_0\rangle$. If it is one of a set of states that are related to one another by T_1 transformations, and which will be denoted $|V_M\rangle$, with M any integer. These are defined by

$$|V_M\rangle \equiv (T_1)^M|V_0\rangle \,, \tag{4.1}$$

where $(T_1)^{-1} = T_{-1}$. It is straightforward to determine the particle content of the $|V_M\rangle$. Consider, for example, the T_1 transform of

$$b^\dagger_{1/2}b_{1/2}|V_0\rangle = 0\,, \tag{4.2}$$

which is

$$T_1 b^\dagger_{1/2}T_1^{-1}T_1 b_{1/2}T_1^{-1}T_1|V_0\rangle = 0\,. \tag{4.3}$$

Using Eq. (2.9) we have

$$d_{1/2}d^\dagger_{1/2}|V_1\rangle = 0 \tag{4.4}$$

which implies

$$d^\dagger_{1/2}d_{1/2}|V_1\rangle = |V_1\rangle \tag{4.5}$$

and therefore $|V_1\rangle$ will have one $d_{1/2}$ background particle. One can show that $|V_1\rangle$ has no other content; $|V_1\rangle \equiv |0;0;1/2\rangle \equiv d^\dagger_{1/2}|0;0;0\rangle$, using the Fock space notation $|\{n_a\};\{n_b\};\{n_d\}\rangle$. Under the R transformation $d \to b$, so that

$$R|V_1\rangle = |V_{-1}\rangle \equiv |0;1/2;0\rangle\,. \tag{4.6}$$

Similar relations hold for the state $|V_M\rangle$ where $-\infty < M < \infty$ and $M < 0$ correspond to states with background b particles.

These states are related by gauge transformations and are therefore "physically equivalent," but for different values of z, since T_1 shifts z by unity. In a given domain, for example $-1 < z < 0$, these states are to be considered as inequivalent. Note that only $|V_0\rangle$ and $|V_1\rangle$ are in the physical subspace as we have defined it; the first is annihilated by Q_3' while the second is annihilated by $1 - Q_3'$.

As discussed previously, we shall use a Schrödinger representation for the gauge degree of freedom described by z and π_z. In this mixed representation, states are written in the form

$$\psi(z)|\{n_a\};\{n_b\};\{n_d\}\rangle\,. \tag{4.7}$$

The object is now to find the lowest-lying eigenstates of the Hamiltonian P^- which are linear combinations of states of this form.

The Hamiltonian is given by

$$P^- = -\frac{g^2}{2(2\pi)^2}\frac{d^2}{dz^2} - g^2\int_{-L}^{L} dx^-\mathrm{Tr}\left(J^+\frac{1}{D_-^2}J^+\right)\,. \tag{4.8}$$

It is convenient to separate P^- into a "free" part and an interaction,

$$P^- = P_0^- + P_I^-\,. \tag{4.9}$$

P_0^- includes all z-dependent c-numbers and one-body Fock operators that arise from normal ordering Eq. (4.8), and has the form

$$P_0^- = C(z) + V(z)\,, \tag{4.10}$$

where $C(z)$ is a c-number function of z and

$$V(z) = \sum_n \left(A_n(z)a_n^\dagger a_n + B_n(z)b_n^\dagger b_n + D_n(z)d_n^\dagger d_n\right)\,. \tag{4.11}$$

The explicit forms of $C(z)$, $A_n(z)$, $B_n(z)$, and $D_n(z)$ are given in the Appendix. P_I^- is a normal-ordered two body interaction. We do not display it here as it is unnecessary for our present purposes. Note that P_0^- itself is invariant under T_1 and R:

$$T_1P_0^-T_1^{-1} = P_0^-(z)\,,\quad RP_0^-R^{-1} = P_0^-\,. \tag{4.12}$$

This is not true of $C(z)$ and $V(z)$ individually.

Consider a possible vacuum state $\zeta(z)|V_0\rangle$, where we choose the fundamental domain $-1 < z < 0$. We consider a matrix element of P^- acting between this state and an arbitrary Fock state. The only non-vanishing matrix element is

$$\langle V_0|P^-\zeta(z)|V_0\rangle = \epsilon_0\zeta(z) \tag{4.13}$$

which leads to Schrödinger equation for $\zeta(z)$:

$$\left[-\frac{g^2}{(2\pi)^2}\frac{d^2}{dz^2} + \frac{g^2}{2}C(z)\right]\zeta(z) = \epsilon_0\zeta(z)\,. \tag{4.14}$$

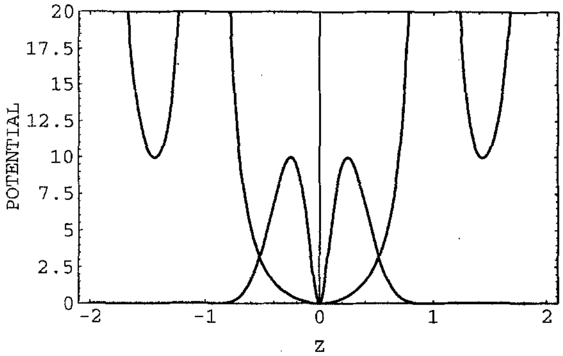

Fig. 1. The potential $C(z)$ as a function of $z = gvL/2\pi$, and the associated wavefunction.

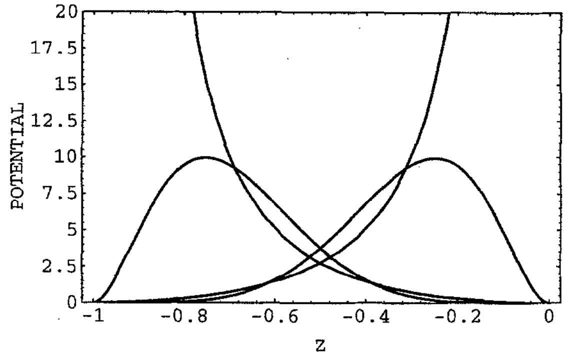

Fig. 2. The potentials $C(z)$ and $C(z+1)$ in the fundamental modular domain and the associated wave functions.

The "potential" $C(z)$ is shown in Fig. 1 and has a minimum at $z = 0$. It is straightforward to solve this quantum mechanics problem with the boundary conditions $\zeta(0) = 0$ and $\zeta(-1) = 0$. These boundary conditions are the result of a number of studies [7,21] of the behavior of states at the boundaries of Gribov regions (in our case, the integer values of z). The shape of the wave function is shown in Fig. 1. To discuss the symmetries of this theory we will find it convenient to define $\zeta(z)$ outside of the fundamental domain. We shall define it to be symmetric about $z = 0$ since $C(z)$ is symmetric about $z = 0$.

Now let us consider the state $\tilde{\zeta}(z)|V_1\rangle$. Projecting the matrix element of $P^-\tilde{\zeta}(z)|V_1\rangle$ with $|V_1\rangle$ we find

$$\left[-\frac{g^2}{(2\pi)^2}\frac{d^2}{dz^2} + \frac{g^2}{2}\left[C(z) + D_{1/2}(z)\right]\right]\tilde{\zeta}(z)$$
$$= \epsilon_1\tilde{\zeta}(z)\,. \tag{4.15}$$

From the explicit forms of $C(z)$ and $D_{1/2}(z)$ given in the Appendix it can be shown that $C(z) + D_{1/2}(z) = C(z+1)$. This is of course just the realization of the T_1 invariance of P_0^-. Setting $\tilde{\zeta}(z) \equiv \zeta(z+1)$ we find the $\epsilon_1 \equiv \epsilon_0$ and the Schrödinger equation is

$$\left[-\frac{g^2}{(2\pi)^2}\frac{d^2}{dz^2} + \frac{g^2}{2}C(z+1)\right]\zeta(z+1)$$
$$= \epsilon_0\zeta(z+1)\,. \tag{4.16}$$

The functions $C(z+1)$, $\zeta(z+1)$, $C(z)$ and $\zeta(z)$ are shown in Fig. 2. From this figure it is clear that $\zeta(z)|V_0\rangle$ and $\zeta(z+1)|V_1\rangle$ are two degenerate vacuum states in the domain $-1 < z < 0$. We have found what we believe are the two expected degenerate vacua for this theory in the domain $-1 < z < 0$. From Fig. 2 we see that these states are reminiscent of the equal-time result [10]. There are some significant differences with the equal-time results however. On the light-front we have an exact expression for the vacuum states, while in the equal-time approach this could only be determined approximately.

The action of the operator T_1R on one of the states $|V_M\rangle$ has strictly speaking only been defined up to a phase,

$$T_1R|V_M\rangle = e^{i\theta_M}|V_{1-M}\rangle\,. \tag{4.17}$$

The phase θ_M is arbitrary, except that it must satisfy

$$e^{i\theta_M}e^{i\theta_{1-M}} = 1\,, \tag{4.18}$$

which follows from $(T_1R)^2 = 1$. We can now construct a vacuum state that is phase-invariant under the symmetry T_1R by superposing our two "n-vacua":

$$|\theta\rangle = \frac{1}{\sqrt{2}}[\zeta(z)|V_0\rangle + e^{i\theta_0}\zeta(z+1)|V_1\rangle]\,, \tag{4.19}$$

where $\zeta(z)$ is normalized to one. It is typically necessary to build the theory on such a vacuum state in order to satisfy the requirements of cluster decomposition.

Finally we would like to briefly discuss P^+. We use an explicitly gauge-invariant form of P^+

$$P^+ = -i\sqrt{2}\int_{-L}^{L} dx^-\mathrm{Tr}\left(\Psi_R D_-\Psi_R\right)\,. \tag{4.20}$$

This is a singular operator and requires regularization and renormalization.

We have done this in two ways, using the gauge-invariant point splitting discussed earlier and also a ζ-function regularization. Both procedures give the same result:

$$P^+ = \frac{\pi}{L}\sum_n n(a_n^\dagger a_n + b_n^\dagger b_n + d_n^\dagger d_n) + \frac{\pi}{L} z Q_3' + \frac{\pi}{2L} z^2 . \tag{4.21}$$

One can explicitly show that this expression is T_1 and R invariant (of course, the exact form of the non-standard terms is essential for this result). The Poincaré algebra here is essentially $[P^-, P^+] = 0$. Explicit calculation gives $[P^-, P^+] = \frac{i\pi}{L}\Big(2\pi_z Q_3' - (z\pi_z + \pi_z z)\Big)$. However the matrix element of the commutator with all physical states vanishes. Thus the Poincaré algebra is valid in physical subspace. This result rests on the fact that we only use the ground state wave function $\zeta(z)$ to construct physical states. There are higher-energy solutions to the quantum mechanics problem in z; however, the energy differences are proportional to L since these energy levels are associated with quantized flux loops that circulate around the closed x^- space. The spectrum of states associated with these very high-energy states decouple in the continuum limit and can be ignored.

5. The condensate

It is generally accepted that QCD in 1+1 dimensions coupled to adjoint fermions develops a condensate. So far this condensate has only been calculated in various approximations. For the vector-like theory it has been calculated in the large-N_c limit in Ref. [8], at high temperature in [9], and in the small-volume limit for $SU(2)$ in Ref. [10]. The theory we are considering here is a chiral theory with only right-handed fermions. It is natural to consider such a theory in a light-front quantized theory because the light-front projections Eq. (2.3) naturally separate the left- and right-handed parts of the theory. Since the theory considered in Refs. [8,10] has both dynamical left- and right-handed fields we do not expect to obtain the same result as those calculations.

The two vacuum states $\zeta(z)|V_0\rangle$ and $\zeta(z+1)|V_1\rangle$ of our chiral theory are both exact ground states. Since we only have right-handed dynamical fermions we only have a spectral flow associated with the right-handed operators, and thus the two physical spaces in the fundamental domain therefore differ by a single fermion. They effectively block diagonalize P^- into two non-communicating sectors. One sector is built on a vacuum with no background particles and the other built on a state with one background particle. Therefore the matrix element of any color singlet operator between these two sectors is expected to vanish. While this theory will not generate a fundamental color-singlet condensate, it does develop a vacuum expectation value for the fermion field.

This is consistent with the equal-time results [10]. We find one type of particle is involved in the spectral flow because there is only a right-handed field, while for the chirally symmetric theory there are two types of particle involved in the spectral flow for $SU(2)$ and four for $SU(3)$. This is directly reflected in the structure of the condensate: for chiral $SU(2)$ the theory develops a condensate for Ψ, while for the vector-like theory a condensate arises for $\bar{\Psi}\Psi$ for $SU(2)$ and for $(\bar{\Psi}\Psi)^2$ for $SU(3)$.

It is straightforward to calculate the vacuum expectation value from Eq. (4.19):

$$\int_{-L}^{L} dx^- \langle\theta|\,\phi_R(x^-)\,|\theta\rangle = e^{-i\theta}\frac{\sqrt{L}}{\pi}\int_{-1}^{0} dz\,\zeta(z+1)\zeta(z) . \tag{5.1}$$

Since Eq. (4.19) is an exact expression for the vacuum, Eq. (5.1) is an exact expression for the vacuum expectation value.

6. Conclusions

We have shown that in QCD coupled to chiral adjoint fermions in two dimensions the light-front vacuum is two-fold degenerate as one would expect on general grounds. The source of this degeneracy is quite simple. Because of the existence of Gribov copies, the one gauge degree of freedom, the zero mode of A^+, must be restricted to a finite domain. The domains of this variable, which after normalization we call z, are bounded by the integers. Furthermore there is a symmetry of the theory under reflections about the midpoint of the fundamental domain. Thus the potential of the vacuum state in the variable z can either have a minimum at $z = 1/2$ or have multiple minima. It has recently been seen that for adjoint scalars the minimum is at $z = 1/2$. In the problem with adjoint fermions described here there are two minima at the ends of the domain.

In the light-front formalism we obtain an exact expression for the vacuum states and we can solve for their fermionic content exactly. We find that these states are very different. They differ because the T_1 transformation gives rise to a spectral flow for the right-handed fermion; thus the two vacuum states differ in the background fermion number and color that each carries. We form the analog of a θ vacuum from these two-fold degenerate vacuum states which respects all of the symmetries of the theory. We find that field ϕ_R has a vacuum expectation value with respect to this θ vacuum and we find an exact expression for this vacuum expectation value.

It is of interest to study whether this Z_N vacuum structure has any effect on observable properties of the theory such as the spectrum of massive states. That is, do the masses depend on the parameter θ? For the vector-like theory this seems unlikely, because the theory with adjoint fermions has the same massive spectrum as some theory with only fundamental fermions [17], where there is no hidden vacuum parameter. For the chiral model discussed here the answer to this question is unknown.

This chiral theory differs from the theories that have been studied in the equal-time formulation [8–10], because the equal-time theory has both dynamical left- and right-handed fields. We expect that if we were to couple together two light-front theories, one with dynamical right-handed particles and the other with dynamical left-handed particles, then the resulting degenerate vacuum and condensate would be exactly calculable and similar to those discussed in the equal-time theory. We shall discuss the details of such a theory elsewhere.

Acknowledgments

It is a pleasure to thank K. Harada for many helpful discussions. S.S.P. would like to thank the Max-Planck-Institut für Kernphysik, Heidelberg, for their hospitality during portions of this work. This work was supported in part by a grant from the US Department of Energy. In addition, D.G.R. was supported during the early stages of this work by the National Science Foundation under Grants Nos. PHY-9203145, PHY-9258270, and PHY-9207889. Travel support was provided in part by a NATO Collaborative Grant.

Appendix A. The gauge potential

We list here the explicit forms of the functions C, A_n, B_n, and D_n discussed in Section 4. The c-number function $C(z)$ must be retained here since it is an operator in z space. The divergence is easily seen to be a true constant and therefore can be subtracted.

$$C(z) = \frac{1}{4}\sum_{n,m}\left[\frac{1}{(n+m+z)^2} + \frac{1}{(m+n-z)^2}\right] - \frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{P}, \tag{A.1}$$

$$A_n = \frac{1}{4}\sum_{m}\left[\frac{1}{(n-m-z)^2} + \frac{1}{(n-m+z)^2} - \frac{1}{(n-m+z)^2} - \frac{1}{(n+m-z)^2}\right], \tag{A.2}$$

$$B_n = \frac{1}{4}\sum_{m}\left[\frac{1}{(m-n+z)^2} - \frac{1}{(n+m-z)^2}\right], \tag{A.3}$$

$$D_n = \sum_{m}\left[\frac{1}{(m-n-z)^2} - \frac{1}{(n+m+z)^2}\right]. \tag{A.4}$$

References

[1] P.A.M. Dirac, Rev. Mod. Phys. 21 (1949) 392.
[2] H.-C. Pauli and S.J. Brodsky, Phys. Rev. D 32 (1985) 1993, 2001.
[3] T. Heinzl, S. Krusche, S. Simbürger and E. Werner, Z. Phys. C 56 (1992) 415;

D.G. Robertson, Phys. Rev. D 47 (1993) 2549;
C.M. Bender, S.S. Pinsky, B. van de Sande, Phys. Rev. D 48 (1993) 816;
S.S. Pinsky and B. van de Sande, Phys. Rev. D 49 (1994) 2001;
J. Hiller, S.S. Pinsky and B. van de Sande, Phys. Rev. D 51 (1995) 726.
[4] A.C. Kalloniatis, H.-C. Pauli and S.S. Pinsky, Phys. Rev. D 50 (1994) 6633.
[5] S.S. Pinsky, in Theory of Hadrons and Light-Front QCD (World Scientific, 1995).
[6] T. Maskawa and K. Yamawaki, Prog. Theor. Phys. 56 (1976) 270;
T. Heinzl, S. Krusche and E. Werner, Phys. Lett. B 256 (1991) 55;
G. McCartor and D.G. Robertson, Z. Phys. C 53 (1992) 679.
[7] J.E. Hetrick, Nucl. Phys. (Proc. Suppl.) 30 (1993) 228.
[8] S. Dalley and I.R. Klebanov, Phys. Rev. D 47 (1993) 2517;
G. Bhanot, K. Demeterfi and I.R. Klebanov, Phys. Rev. D 48 (1993) 4980;
I.I. Kogan and A.R. Zhitnitsky, OUTP-95-25 P, hep-th/9509322.
[9] A. Smilga, Phys. Rev. D 49 (1994) 6836.
[10] F. Lenz, M. Shifman and M. Thies, Phys. Rev. D 51 (1995) 7060.
[11] E. Witten, Nuovo Cim. A 51 (1979) 325.
[12] G. McCartor, Z. Phys. C 41 (1988) 271.
[13] L.D. Faddeev, Phys. Lett B 14 (1984) 649;
R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219;
R. Rajaraman, Phys. Lett. B 154 (1985) 305;
L.D. Faddeev and S.L. Shatashvili, Phys. Lett. B 167 (1986) 225;
E. Abdalla and M.C.B. Abdalla, hep-th/9503002, to appear in Physics Reports;
F.M. Saradzhev, hep-th/9601100.
[14] K. Demeterfi, I.R. Klebanov and G. Bhanot, Nucl. Phys. B 418 (1994) 15;
H.-C. Pauli, A.C. Kalloniatis and S.S. Pinsky, Phys. Rev. D 52 (1995) 1176;
S.S. Pinsky and A.C. Kalloniatis, Ohio State University preprint OHSTPY-HEP-TH-95-014, hep-th/9509036, to appear in Phys. Lett. B;
A.C. Kalloniatis, hep-th/9509027.
[15] A.C. Kalloniatis and D.G. Robertson, Phys. Rev. D 50 (1994) 5262.
[16] B. van de Sande and M. Burkardt, hep-th/9510104, to appear in Phys. Rev. D;
F. Antonuccio and S. Dalley hep-lat/9505009; hep-ph/9506456.
[17] D. Kutasov and A. Schwimmer, Nucl. Phys. B 442 (1995) 447.
[18] D.J. Gross, I.R. Klebanov, A.V. Matytsin and A. Smilga, hep-th/9511104.
[19] V.A. Franke, Yu.A. Novozhilov and E.V. Prokhvatilov, Lett. Math. Phys. 5 (1981) 239, 437; and in: Dynamical Systems and Microphysics (Academic Press, 1982).
[20] V.N. Gribov, Nucl. Phys. B 139 (1978) 1.
[21] F. Lenz, H.W.L. Naus and M. Thies, Ann. Phys. (N.Y.) 233 (1994) 317;
M. Engelhardt and B. Schreiber, Z. Phys. A 351 (1995) 71.