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Abstract

Maximum likelihood estimation of the two-parameter Weibull distribution is straightforward; however, there are multiple methods for maximum likelihood estimation of the three-parameter Weibull. This paper presents an evaluation of these methods using four data sets including oil spill data from the Gulf of Mexico. Highlighted are fairly major differences in the estimated parameters between nine statistical packages. A VBA routine has been developed allowing practitioners to implement three-parameter Weibull maximum likelihood estimation within Excel. The code and supporting documentation are available free at http://faculty.otterbein.edu/WHarper.

Key Words: Weibull, Maximum Likelihood, Reliability

1. Introduction

When performing a statistical test or building a statistical model, the analyst expects the statistical results to be independent of the software package. For example, in linear regression one anticipates obtaining the same equation independent of the software package. Additional options may vary from one package to another especially with items such as regression diagnostics and graphics. In a similar vein for doing maximum likelihood estimation of the parameters of a given distribution, one might expect different goodness of fit tests (e.g., chi-square, Kolmorgov-Smirnoff, Cramer von-Mises, Anderson-Darling), and graphical output (e.g., probability plots, empirical distribution functions, P-P plots, Q-Q plots). But one expects the same parameter estimates within rounding.

Such is not the case for estimation of the 3-parameter Weibull. The 3-parameter Weibull has been documented as a challenge when finding maximum likelihood estimates (MLEs). This article highlights fairly major differences in estimated parameters between nine statistical packages.

Our research began with the use of Minitab for distribution fitting related to oil spill data in the Gulf of Mexico. Driving part of recent efforts was a desire to develop an Excel VBA routine for the 3-parameter Weibull. In using additional statistical software packages, it was noted that the MLEs varied more than anticipated. This led to an extensive literature search, some of which is summarized in the literature review section, as well as the use of numerous software packages. The results of the investigation are documented in this report.

An initial 3-parameter Weibull Excel VBA code is available free at http://faculty.otterbein.edu/WHarper/. This Excel VBA code is not intended to be a complete package or replacement for other well developed Weibull statistical software. Its primary purpose is to allow some Weibull MLE analyses within the context of Excel. One of the longer range goals is to provide Weibull percentile confidence intervals. One should still use another statistical package for goodness of fit criteria and other graphical options available before assuming a Weibull distribution is reasonable for a given application.

This paper is organized as follows. A brief literature review summary is provided in Section 2. Section 3 provides background information on the oil spill data in the Gulf of Mexico that represents two of the four data sets used in this paper. Section 4 lists all four data sets used along with our initial limited use of more than one software package for MLEs of the 3-parameter Weibull distribution. General comments on Weibull estimation are found in Section 5. Section 6 lists the nine different software packages and summary Weibull MLE results for each of the four data sets as well as an overview of different general maximum likelihood estimation approaches for the 3-parameter Weibull. Finally the summary and conclusions are in Section 7.

2. Selected Literature Review

Extensive literature reviews of the Weibull are available in a variety of places. This section highlights some of the literature we found to be interesting. These and other sources are cited as needed later in this paper.

King (1981) discusses the uses of Weibull probability paper for the 3-parameter Weibull distribution. While many today will no longer manually use probability paper, understanding the mechanics of using Weibull probability paper aids in the understanding of the probability paper based methods found in specialized reliability software packages.

One of the challenges of a literature search is keeping track of both the Weibull parameter notation and the terminology. Below are both the pdf and cdf that we are using.

$$pdf,\ f(x) = \beta\alpha^{-\beta}(x-\gamma)^{\beta-1}e^{-((x-\gamma)/\alpha)^{\beta}} \ \ for\, x > \gamma;\, 0\ otherwise.$$
$$cdf,\ F(x) = 1 - e^{-((x-\gamma)/\alpha)^{\beta}}$$

Zanakis (1979) used different Weibull notation. In our notation he points out that γ, the 3rd Weibull parameter goes by a variety of names such as location, minimum life, threshold, and origin. From other sources: terms such as guaranteed minimum life, guaranteed life, and shift are common. α is generally called the scale and β is either shape or slope (typically in probability paper or rank regression based approaches). Zanakis (1979) provides a Monte Carlo comparison of 17 different methods including the method of moments, MLE, and least squares. These methods are compared in a mean square error sense.

Sen and Prabhashanker (1980) created a nomogram to estimate all 3-parameters using the method of moments. As they along with others correctly point out the desirable properties of maximum likelihood estimation are based on asymptotic results which often will not be attained in practice with small sample sizes.

Smith and Naylor (1987) make a strong pitch for a Bayesian approach. They note that the Bayesian methodology does not rely on asymptotics. They admit that the problem of the selection of a prior distribution exists but argue that the resulting sensitivity analysis that can be performed for different priors is informative and worth the effort.

Hirose (1996) points out that the non-regular cases occur when the shape parameter β is ≤ 2. There are no MLEs when $0 < \beta < 1$, and the MLEs exist but are not asymptotically normal when $1 \leq \beta \leq 2$. Additionally MLEs may encounter parameter divergence. Hirose recommends the Generalized Extreme Value (GEV) distribution as a better estimation approach for the 3-parameter Weibull. He shows that a more stable search occurs in the GEV space than the Weibull space.

Johnson et al (2004) state that MLEs are only regular for $\beta > 2$. If β is in the interval (0, 1) then the min $X_{(1)}$ is superefficient (called hyper-efficient in other articles).

Lawless (2003) describes what seems to be the most common approach and the one we take in our VBA code. Since finding the general 3-parameter MLEs is difficult, Lawless suggests maximizing the profile log-likelihood function. A profile log-likelihood fixes one or more parameters of the distribution and then maximizes the log-likelihood for the remaining parameters. This is performed for the 3-parameter Weibull by assessing potential values for the threshold and then maximizing the log-likelihood at this fixed threshold. This in essence is just solving the two parameter MLE problem for $x_i^* = x_i - \hat{\gamma}$ where x_i, $\hat{\gamma}$ are the original data values and current value of the threshold, respectively.

Zanakis (1977) examines 7 different algorithms for 225 test problems. From Zanakis and other sources, MLE issues may be broken into the following intervals for the shape parameter β:

1. $\beta \le 1$: here the smallest observation becomes a hyper-efficient solution for γ.
2. $1 < \beta < 2$: MLEs exist but the asymptotic covariance matrix is meaningless yielding negative variances.
3. $\beta = 2$ (Rayleigh distribution): determinant of the information matrix vanishes.

3. Background of Platform and Pipeline Oil Spill data in the Gulf of Mexico

Previous analyses of oil spills in the Gulf of Mexico (Anderson & LaBelle, 2000) focused on oil spills of at least 1,000 barrels. Spills were categorized as either platform spills or pipeline spills. Eschenbach & Harper (2006) examined spills of at least 50 barrels for the Minerals Management Service (MMS) of the U.S. Department of the Interior. This increased the number of observations in the data sets and was felt to provide a more comprehensive assessment of items of interest such as the likelihood of very large spills, and the stationarity of oil spill rates over time in their 2006 report. A database is not kept for oil spills less than 50 barrels.

As part of this project, Eschenbach & Harper fit statistical distributions to oil spill volume data. In all cases the data was left censored with varying cutoffs on the low end ranging from 50 barrels to 1,000 barrels. This analysis was done with Minitab. Pipeline oil spill data was found to be well fit by a by a 3-parameter Weibull distribution while platform spill volume data was better fit by a 3-parameter lognormal distribution. A 3-parameter Weibull fit to the platform spill data did not pass the Anderson-Darling goodness of fit test used in Minitab but was a better visual fit than most other distributions.

Both platform and pipeline spill data sets plus two more data sets described later were used in this paper to assess 3-parameter Weibull MLEs.

4. Initial Limited Check on other Weibull Software

The scope of the MMS work was broad and many topics were investigated, but as with many projects there were additional future areas for subsequent study and follow-on. As the boundaries were pushed to predict larger spills, Minitab did not help answer important questions. The few other initial software packages available did help some, but it was noticed that the 3-parameter Weibull distribution estimates varied more than anticipated from one software package to another. This furthered the literature search and the desire to develop Excel VBA code. Many statistical software vendors were kind and lent their packages to help in this exploration. The totally free software is sincerely appreciated as well as all the valuable help and sharing of references. Every vendor dealt with was friendly and professional. While the full results to be published later show some vendors had difficulties with certain data sets, it is definitely not the intent to downgrade or bash any software package. Without the support of the various statistical software specialists, it would not have been able to learn as much.

The 2-parameter Weibull (shape and scale) is not an estimation problem for any statistical package encountered. Indeed the initial VBA code was quickly developed, tested, and verified for the 2-parameter case.

The problem comes with the third parameter which goes by various names such as shift or threshold. Instead of the Weibull starting at 0, the third parameter shifts the origin right or left though more commonly to the right to some positive value. This is sometimes called the minimum guaranteed life in reliability analysis in which time is often the driving force. As a slight aside the driving force may be thought of as an exposure variable in which the likelihood of survival depends on the amount of the exposure variable an item (say a part) has experienced. Bearings for example are often modeled as having some positive threshold or guaranteed minimum number of revolutions before failure might occur.

In the context of our oil spill data the third parameter shifts the distribution to the right, since the data and the various spill size models have minimum spill size thresholds of 50 to 1000 barrels.

Lockhart and Stephens (1994) was a reference given in the Minitab help documenting their methodology. Lockhart and Stephens include two data sets in their publication. For this study four data sets are used: MMS platform oil spill data; MMS pipeline oil spill data; and the two Lockhart and Stephens data sets (LS Ex1; LS Ex2 below). To allow others to examine this problem, all four data sets are listed in Table 1 and are also available along with our VBA software at http://faculty.otterbein.edu/WHarper/.

Table 1: Four Data Sets Used In The 3-Parameter Weibull Fitting.

LS Ex1 $n = 10$	LS Ex2 $n = 15$	Pipeline Spill Volumes ≥ 50 $n = 36$		Platform Spill Volumes ≥ 50 $n = 78$				
117	12	50	800	50	75	107	228	1500
135	21	51	900	51	76	118	239	1588
135	26	52	1211	52	77	119	258	2225
162	27	53	2000	53	78	120	264	7000
162	29	65	2240	54	79	125	280	7371
171	29	80	3200	55	80	126	286	9935
189	48	81	3445	56	83	127	300	
189	57	100	3500	57	89	130	320	
198	59	101	4000	58	95	148	321	
225	70	119	4533	59	96	150	400	
	74	135	4569	60	97	165	401	
	153	190	5000	61	100	166	430	
	326	210	5100	62	101	170	435	
	386	250	8162	63	102	200	600	
	502	300	8212	64	103	201	601	
		323	14423	65	104	202	643	
		414	15576	66	105	210	1053	
		553	19833	67	106	214	1456	

5. General Comments on Weibull Estimation

One of the interesting aspects of the work behind this paper was realizing that the major statistical packages generally offer only maximum likelihood estimates for distribution fitting. This had not been a concern before as estimation problems were generally non-existent until this 3-parameter Weibull situation. The literature discusses other approaches, but few statisticians today will do manual computation or break out graphing paper. Instead computer packages are used to estimate parameters, evaluate the choice of distributions, and graph the results.

Fortunately specialized reliability software packages incorporate multiple estimation methodologies (e.g., median rank regression) that are both simple to perform and allow a quick comparison of differing estimates based on the method chosen. In this paper it is not possible to fully explore such alternative options to maximum likelihood estimation. We'd like to note that it is both educational and worthwhile to get back in touch with these methods which some may have grown up with prior to maximum likelihood estimation being feasible with computers.

Maximum likelihood has much to offer (consistency, asymptotic normality, and asymptotic efficiency) but such properties are based on large samples. In practice large samples may not be available and thus the often stated advantages of maximum likelihood estimation are not applicable to moderate sized samples such as the four data sets in this study.

Minitab and Palisade's Best-Fit use approaches based on Lockhart and Stephens. Our current VBA code is based in part on it also. It is important when examining the summary software package results to not mentally punish vendors that take somewhat different approaches to handling the 3-parameter Weibull as there are other approaches used to estimate the threshold parameter beyond what is found in Lockhart and Stephens.

It was anticipated that Minitab would replicate the results in Lockhart and Stephens. Minitab did for their first data set but not for their second. Checking with Minitab, it was discovered that Minitab (Banga, 2005) had improved upon an approximation used in Lockhart and Stephens. This improvement for an unbiased estimate of the threshold is also in the Mann, Schafer, and Singpurwalla (1974) text which in turn references the original work in Dubey (1966). This is also included in our VBA routine.

Table 2 presents summary 3-parameter Weibull distribution fitting results for 11 methods from 9 statistical packages. The 9 packages are JMP, SAS, SPSS, Minitab, Palisade's Best-Fit, Arena's Input Analyzer, Dodson's Weibull, Weibull++7, and WinSmith Weibull. SPSS offers just the 2-parameter Weibull and thus does not provide solutions to any of the four data sets. This is not counted as a failure in Table 2.

Table 2: Maximum Likelihood Parameter Estimates For The 3-Parameter Weibull.

	# Failed	Min	Max	Average	Median
Lockhart & Stephens 1 (n=10)					
Threshold	1	85.8	117.0	101.4	99.0
Shape	1	0.8	2.9	2.2	2.4
Scale	1	48.1	92.7	75.1	78.2
Lockhart & Stephens 2 (n=15)					
Threshold	2	8.7	12.0	11.2	12.0
Shape	2	0.6	0.8	0.7	0.7
Scale	2	76.1	101.0	85.3	83.0
Pipeline Spills (n=36)					
Threshold	1	-5258.2	50.0	-540.1	50.0
Shape	1	0.4	1.9	0.6	0.5
Scale	1	1370.0	9444.2	2332.8	1448.0
Platform Spills (n=78)					
Threshold	2	49.5	50.0	49.9	50.0
Shape	2	0.5	0.5	0.5	0.5
Scale	2	193.0	205.5	197.3	196.2

Not only do different packages give different estimated parameter values, but whether or not a solution can be found varies as well. For each of the four data sets some packages failed to find a solution, and different packages fail on different data sets. In addition the task of estimating the threshold introduces variability into the estimated values for the shape and scale.

We do not know whether the Lockhart & Stephens data was chosen because it was challenging for maximum likelihood estimation, but the MMS oil spill data is real and the challenges arose while we were engaged in a genuine estimation situation.

JMP, SAS, SPSS, and Minitab are well established general purpose statistical packages. Palisade's Best Fit is a well known distribution fitting package. Arena's Input Analyzer has not been updated in years but is the distribution fitting software that comes bundled with the popular Arena simulation software. The remaining entries are specialized packages that focus on reliability. Dodson's Weibull comes bundled with his book (Dodson, 2006). Weibull++7 is developed by Reliasoft and has a book (ReliaSoft, 2005) available for either purchase in hard copy or downloadable free from the web. Weibull++7 is part of a suite of reliability based software. WinSmith Weibull is a statistical reliability package and is the featured statistics package in the enjoyable *The New Weibull Handbook* (Abernethy, 2006).

6. Approaches to Maximum Likelihood Estimation for the Weibull

There are varieties of ways to develop the maximum likelihood estimates for a 3-parameter Weibull. Most of the packages evaluated follow the three steps below for maximum likelihood estimation for the 3-parameter Weibull (the 3 exceptions are SAS, JMP, and one of the Weibull++7 methods).

1. Estimate the threshold parameter γ. This is done in many different ways.
2. Subtract the estimated threshold (or threshold – ε) from the values in the data set.
3. Do 2-parameter Weibull maximum likelihood estimation to obtain the other 2 parameters using all n observations from step 2.

The variability in the results is often due to how the threshold parameter is estimated.

Some minimal experimentation is needed as the vendor's value ε is generally not documented. It is worthwhile to go through this exercise as it helps learn what the vendors are doing as well as finding out exceptions to the above. Our personal feeling is that there is not just one correct way to estimate the threshold γ; therefore, results exhibiting moderate variability from another are fine in our opinion. It is also instructive to learn more about how and why a given method works the way it does.

Palisade's Best-Fit uses a Lockhart & Stephens based approach but only when the shape parameter β is estimated to be > 1. For the other packages that follow the 3 steps above, it is interesting to see the variability in the estimates.

JMP has an unusual procedure in which the smallest data value is set to the threshold and subtracted from the other observations. Then the smallest observation is dropped from the data set so that the $n - 1$ observations left are fit with a 2-parameter Weibull. SAS and "Weibull++7 True 3 parm." are the only methods that attempt a true simultaneous maximum likelihood estimation of all three parameters. This has been historically documented to be an extremely difficult problem to solve for many reasons given in the literature. For these four data sets, SAS only estimates the parameters for the Pipeline Spills data set. Unfortunately these estimates are unreasonable for that data set.

"Weibull++7 True 3 parm." takes advantage of the relationship between the generalized extreme-value distribution and the Weibull (Hirose, 1996) and seems to be successful in obtaining reasonable results for a difficult to solve optimization problem. It should be noted that Reliasoft (personal telephone conversation with Reliasoft's David J. Groebel on February 22, 2008) does not recommend solving the problem using this approach as it feels its other options are more practical such as least squares rank regression. Furthermore, page 546 of Reliasoft (2005) documents the two problems of non-regularity and parameter divergence when doing the true three parameter maximum likelihood estimation for the Weibull.

The VBA for Excel code follows the 3 steps listed near the beginning of this section. It allows the user to specify the threshold parameter γ if desired. If no threshold parameter is input, then the algorithm follows a procedure based on the Benga (2005) updated version of Lockhart and Stephens (1994). We choose this approach for two reasons: 1) we agree with the statement made by Panchang, and Gupta (1989) that the procedure outlined by Lawless (latest edition, 2003) "is the only one that is guaranteed to yield parameter estimates that maximize the likelihood function for any sample.", and 2) it was felt that Lockhart and Stephens is a reasonable (but not the only) approach to solving this problem as it also accounts for the bias of the maximum likelihood threshold parameter estimate. More detailed documentation is bundled in the zip file that includes the Excel VBA code

7. Summary and Conclusions

This paper documents some challenges of estimating the 3-parameter Weibull distribution using maximum likelihood estimation. It shows that considerable variability exists in results reported by different statistical packages. These differences may be critical for those who would use the results. Because this paper is based on only 4 examples firm conclusions about the relative value of the different packages are clearly inappropriate. Different conclusions may be reached for other distributions and other data sets.

This paper adds worthwhile detail and useful examples to the literature on the challenges of estimating the 3-parameter Weibull distribution with maximum likelihood. It also presents a free VBA implemented approach that can be used with Excel for those who need to estimate the 3-parameter Weibull.

As mentioned earlier, maximum likelihood is not the only procedure available when estimating population parameters. For difficult estimation problems especially for non-large sample sizes, alternatives such as found in WinSmith Weibull, Weibull++7, and Dodson's Weibull are worthy of serious investigation.

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