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ABSTRACT: Light-cone quantization of gauge field theory is considered. With a careful treatment of the relevant degrees of freedom and where they must be initialized, the results obtained in equal-time quantization are recovered, in particular the Mandelstam-Leibbrandt form of the gauge field propagator. Some aspects of the "discretized" light-cone quantization of gauge fields are discussed.

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1. Introduction

When setting up a light-cone quantized version of a gauge theory it is most natural to choose the light-cone gauge: $n \cdot A = 0$, with $n^2 = 0$. This choice is particularly convenient when $n \cdot x$ is the evolution parameter of the system. (We shall define $x^{\pm} \equiv x^0 \pm x^3$, and take x^+ to be the evolution parameter. Thus we take $n_\mu = (1, 0, 0, 1)$.) In this case Gauss' law appears to be a constraint relation, which can be solved for A^- in terms of the transverse degrees of freedom. Thus one can eliminate all unphysical components of the gauge field at the level of the Hamiltonian. The Fadeev-Popov sector decouples, so that unitarity is manifest. And finally, the light-cone gauge, like other algebraic noncovariant gauges, is free of the Gribov ambiguity.

However, implementation of this gauge is not entirely straightforward, at least in perturbation theory. The problem is easily stated: the naive gauge field propagator

$$D^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k}\right] \tag{1.1}$$

is singular at $n \cdot k = k^0 + k^3 = 0$. (In a Hamiltonian framework this appears as singularities in certain matrix elements of the Hamiltonian.) The issue is how this object is to be given a meaning.

Various suggestions have been made in this regard. The simplest approach is to interpret the singularity as a Cauchy principal value [1]. This can easily be seen to be wrong, however, on physical grounds. The basic problem is that the spurious poles lie in different quadrants of the complex k^0-plane than do the usual Feynman poles. Thus positive (negative) energy quanta do not necessarily propagate into the forward (backward) light cone. Practically speaking, what happens is that a Wick rotation is impossible without crossing a spurious pole, and extra contributions are generated. These extra terms are responsible for the well-documented failure of calculations performed in this way to agree with calculations performed in, *e.g.*, covariant gauges [2].

A more successful prescription for the spurious singularity is that suggested independently by Mandelstam [3] and Leibbrandt [4] (ML). They define (in Leibbrandt's notation)

$$\frac{1}{[k^+]_{\rm ML}} \equiv \frac{k^-}{k^+ k^- + i\epsilon}, \tag{1.2}$$

so that the pole in k^+ is shifted above or below the real axis depending on the sign of k^-. With this definition, the spurious poles are distributed in the same way as the Feynman poles; thus no poles are encountered in performing a Wick rotation, and extra terms are not generated. Many explicit calculations have been performed using the ML prescription, and all give sensible results, in agreement, where comparison is possible, with covariant-gauge results.

Bassetto, *et al.* have given a derivation of the ML propagator in the framework of equal-time quantization [5], and have further shown that gauge theories defined in this way are renormalizable [6] (although nonlocal counterterms are necessary to render off-shell Green functions finite). A central feature of their construction is that they do not reduce down to the physical (transverse) degrees of freedom. A longitudinal gauge degree of freedom is retained, and a corresponding ghost field. In a way familiar from covariant-gauge quantization, selection of a physical subspace results in the recovery of a positive-semidefinite metric and Poincaré invariance. More concretely, one considers the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \lambda^a n_\mu A^{\mu a}, \tag{1.3}$$

with λ^a a Lagrange multiplier field whose equation of motion enforces the gauge condition $A^{+a} = 0$. In addition to λ^a, one keeps the other three components of the gauge field as degrees of freedom. Equal-time quantization of this Lagrangian leads unambiguously to the ML form of the propagator. It is an open—and interesting—question whether a formulation exists in which only physical degrees of freedom are retained and the gluon propagator is well defined.

Light-cone quantization of the Lagrangian (1.3), however, is slightly mysterious. The puzzle has to do with the field λ^a; it satisfies

$$\partial_- \lambda^a = 0, \tag{1.4}$$

which is an equation of constraint on the light-cone initial-value surface $x^+ = 0$. Thus if we impose the natural boundary condition that λ^a vanish at $x^- = \pm\infty$, we obtain $\lambda^a = 0$ everywhere, and a formalism that cannot be equivalent to the equal-time quantized version of (1.3). In particular, we are led to a principal value prescription (or equivalent) for the spurious pole.

The purpose of the present paper is to show how a careful light-cone quantization of the Lagrangian (1.3) in fact leads to precisely the results obtained by Bassetto, *et al.*, and in particular to the ML form of the gauge field propagator. The main point we shall make is that certain of the degrees of freedom—including the field λ^a—must be initialized along a surface of constant x^-, in a way familiar from the treatment of massless fields in two spacetime dimensions, for example.

We shall begin in Section 2 by identifying the relevant degrees of freedom and how they must be initialized. We then discuss the construction of the dynamical operators $P^\pm$ in the free theory and the determination of the commutation relations to be satisfied by the fields. The free theory is trivially solved, and we compute the propagator. Finally, we show how the physical subspace is to be selected, and the Poincaré invariance of the theory recovered. Section 3 contains some remarks on the problem of quantizing a gauge field with a periodicity condition imposed along a characteristic surface—the case of "discretized" light-cone quantization. Some concluding remarks are presented in Section 4.

Another promising subtraction procedure, inspired by the ML prescription and realizable in the context of light-cone perturbation theory, has recently been proposed [7,8,9]. Its consistency has yet to be explored in detail, however.

2. Light-Cone Quantization of the Lagrangian (1.3)

In this section we show how a careful light-cone quantization of the Lagrangian (1.3) leads to the same results obtained in equal-time quantization [5]. We shall begin by considering the free theory ($g = 0$) and, to avoid spurious complexities, only a single gauge field. That is, we study the problem of QED with no fermions.

Our first task is to use the equations of motion to identify the degrees of freedom of the system, that is, those data that are necessary to allow the most general solution to the equations of motion throughout all of spacetime. These data will correspond to independent operators in the quantum field theory, whose commutation relations are determined by demanding that the Heisenberg equations of motion correctly reproduce the classical field equations, and that the Poincaré algebra be correctly realized. The most straightforward way to proceed is simply to imagine using the field equations to evolve the fields from some assumed classical initial data. If all of spacetime can be filled out in this way, then the assumed data is sufficient. One then tries to uncover redundancies, in order to arrive at the minimal data that allows a completely general solution. These minimal data then correspond to operators which we must include when we go over to the quantum field theory.

The procedure is most easily explained by example. Let us consider the case of a free, massless Dirac field in two spacetime dimensions. As is well known, the two-spinor ψ can be broken up into two one-spinors $\psi_\pm \equiv \frac{1}{2}\gamma^0\gamma^\pm\psi$, corresponding (in two dimensions) to chirality eigenstates. In terms of these fields the Dirac equation separates into

$$\partial_+\psi_+ = 0 \tag{2.1}$$

$$\partial_-\psi_- = 0. \tag{2.2}$$

Now, the solution of eqs. (2.1)–(2.2) is trivial: ψ_+ (ψ_-) can be any arbitrary function of x^- (x^+). In the language of evolving from initial data, we might say that we specify ψ_+ along $x^+ = 0$ and ψ_- along $x^- = 0$. We then use eqs. (2.1)–(2.2) to evolve the fields off of these surfaces; although in this case the evolution is particularly simple. Each field is simply constant in its "evolution parameter." The initial data is clearly sufficient to determine $\psi_\pm$ everywhere, and it is also clear that it corresponds to the most general solution of the free, massless Dirac equation in two dimensions. When we build the quantum theory, then, we must include *both* a ψ_+ field, satisfying an equal-x^+ commutator, and a ψ_- field satisfying an equal-x^- commutator. Throwing away some of these degrees of freedom—setting ψ_- to zero, for example—results in a theory which is not isomorphic to the equal-time theory of a massless Dirac particle [10]. In particular, if we leave out the ψ_- degrees of freedom then the theory will contain only right-moving particles.

One lesson of this example is that it is not always possible when quantizing on characteristic surfaces to have a strictly "Hamiltonian" framework, with all the dynamical variables evolving from a single initial-value surface. In general, degrees of freedom initialized on other characteristics may need to be included. (The unusual feature is not so much that the initial-value surface has a kink in it, as that some variables are initialized on one part while others are initialized on another part.) Particular subsets of the full solution space may be selected by imposing boundary conditions on the fields, but the resulting theory will not generally represent all of the desired physics. In the case of the (1+1)-dimensional massless fermion, for example, we could demand that ψ_- vanish at $x^- = \infty$. This would then force $\psi_- = 0$ everywhere, leading to a *different* theory. Some physics—in this case, the physics of left-moving particles—has been excluded.

Let us now consider the gauge theory. The equations of motion obtained from the Abelian version of (1.3) can be cast in the compact form

$$\partial_\mu F^{\mu\nu} = n^\nu\lambda. \tag{2.3}$$

Writing these out for the different values of ν, in terms of light-cone variables and with A^+ set to zero, we have

$$(\partial_-)^2 A^- + \partial_- \partial_i A^i = 0 \tag{2.4}$$

$$2\partial_+\partial_- A^- - \partial_\perp^2 A^- - 2\partial_+\partial_i A^i = 2\lambda \tag{2.5}$$

$$(4\partial_+\partial_- - \partial_\perp^2)A^i + \partial_i(\partial_- A^- + \partial_j A^j) = 0. \tag{2.6}$$

Note also that, from eq. (2.3), the field λ satisfies

$$n \cdot \partial\lambda = \partial_- \lambda = 0. \tag{2.7}$$

Our first observation is that the solution of eq. (2.7) is simply that λ be an arbitrary function of $(x^+, x_\perp)$. *Thus λ is analogous to the ψ_- field in the example above; it must be initialized along a surface of constant x^-.* Different choices for λ give different solutions to the field equations, so in the quantum theory we must have degrees of freedom corresponding to the freedom to specify λ. These fields will satisfy an equal-x^- commutation relation with some "conjugate momentum" yet to be determined, and their contribution to conserved charges will be obtained by integrating suitable components of the charge densities over their initial-value surface. Note that one could impose, for example, the condition that λ vanish at $x^- = \pm\infty$, leading to $\lambda = 0$ everywhere. As in the previous example, however, this results in the selection of a subspace of the full solution space; the resulting field theory will not contain all the relevant degrees of freedom.

Let us now consider eq. (2.4). This may be solved to yield

$$\partial_- A^- + \partial_i A^i = \varphi(x^+, x_\perp) \tag{2.8}$$

where φ is again arbitrary (note that φ is just $\partial_\mu A^\mu$ in this gauge). This is another function which must be specified to determine the complete solution of the field equations, and will correspond to degrees of freedom in the quantum theory. Again, by imposing a condition on $\partial_- A^-$ at $x^- = \pm\infty$ we could remove this freedom, but the resulting field theory would be incomplete.

The solution of eq. (2.8) apparently involves yet another undetermined function of $(x^+, x_\perp)$:

$$A^- = -\frac{1}{\partial_-}\partial_i A^i + x^-\varphi + \gamma(x^+, x_\perp) \tag{2.9}$$

(where $(\partial_-)^{-1}$ is some particular antiderivative), but we shall see below that there is a constraint relating γ, φ, and λ, so that only two of them are in fact independent. This is an example of an apparent freedom in the system which is actually redundant.

Finally, we must give some initial data for A^i. It is convenient to define

$$A^i \equiv T^i + \frac{\partial_i}{\partial_\perp^2}\varphi, \tag{2.10}$$

so that eq. (2.6) becomes simply

$$(4\partial_+\partial_- - \partial_\perp^2)T^i = 0. \tag{2.11}$$

The initial data appropriate for a massless scalar field in 3+1 dimensions are the values of the field along $x^+ = 0$. The only subtlety concerns the set of modes with $k_\perp$ and k^+ identically zero. These represent quanta propagating precisely along the surface $x^+ = 0$, so that they can not be initialized there. They are somewhat awkward to treat in a continuum formalism, as it is difficult to properly "measure" their contribution. (Note that they are of measure zero even relative to the "zero mode" fields λ and φ.) This is a standard problem in light-cone quantization, which is conventionally treated by choosing the test functions in which to smear the fields to vanish at $k^+ = 0$ [11]. Thus they can be effectively neglected, with the added bonus that the integral operator $(\partial_-)^{-1}$ will be uniquely defined when acting on T^i (it has no $k^+ = 0$ mode).

We have now exhausted eqs. (2.4) and (2.6), and it is clear that from these we can determine the solution for A_μ everywhere in spacetime. All that remains is to insure that the solution thus obtained is consistent with the last equation, (2.5). To this end, we insert the definitions of φ, γ, and T^i (eqs. (2.8), (2.9), and (2.10), respectively) into eq. (2.5). Making use of eq. (2.11), along with the fact that $(\partial_-)^{-1}T^i$ is uniquely defined, we obtain

$$-\partial_\perp^2\gamma - 2\partial_+\varphi = 2\lambda. \tag{2.12}$$

Thus the three "zero mode" fields λ, φ, and γ are not independent, as promised. We shall here take γ to be the determined quantity, and treat λ and φ as independent fields.

Our final result is that the initial data required to determine the most general solution to the classical field equations are: T^i on the traditional light-cone initial-value surface $x^+ = 0$; and any two of λ, φ, and γ on a surface of constant x^-. (Of course, these fields are by definition independent of x^-, so it does not matter exactly where we put them. For reasons which should become clear below, we shall imagine them to be initialized along the boundary "wings" at $x^- = \pm\infty$—surface 2 in Fig. 1.) The next step is to determine the commutation relations satisfied by these fields, by constructing the Poincaré generators and demanding that the Heisenberg equations correctly reproduce the field equations (2.4)–(2.6).

The main complicating feature in the computation of conserved charges is that we must include contributions from the boundary surfaces (Fig. 1). This actually follows quite generally if we insist that the charges we construct be identical to those we would obtain in equal-time quantization [10]. We have

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.13}$$

so that

$$\oint T^{\mu\nu} d\sigma_\mu = 0, \tag{2.14}$$

where the integral is taken over a closed surface. If this surface is taken to be that shown in Fig. 1, then it is clear that the integral over the equal-time surface $t = 0$ is equal to that over the light-cone surface, including the boundary wings. Thus in general we must retain contributions coming from the boundary surfaces to be assured of obtaining correct results. In some cases—for example when only massive fields are present—it may be consistent to discard the boundary contributions. In effect, we can assume that massive fields are sufficiently damped at infinity so that these contributions are negligible. When massless fields are present, however, this is generally not the case.

Let us focus on the generators of translations $P^\pm$. The (Noether) energy-momentum tensor derived from the Lagrangian (1.3) is

$$T^{\mu\nu}_{\rm Noether} = -F^\mu_{\ \sigma} F^{\nu\sigma} + \frac{1}{4} g^{\mu\nu} F^2 - \lambda n^\mu A^\nu - \partial_\sigma (F^{\mu\sigma} A^\nu) \tag{2.15}$$

As usual, this is neither symmetric nor gauge invariant, but satisfies $\partial_\mu T^{\mu\nu}_{\rm Noether} = 0$. By virtue of the antisymmetry of $F^{\mu\nu}$, the four-divergence of the last term in eq. (2.15) itself vanishes; thus the modified tensor

$$T^{\mu\nu} = -F^\mu_{\ \sigma} F^{\nu\sigma} + \frac{1}{4} g^{\mu\nu} F^2 - \lambda n^\mu A^\nu \tag{2.16}$$

will also give time-independent generators. We shall here take (2.16) as the definition of the energy-momentum tensor.

We consider first P^-. We have

$$P^- = \frac{1}{2} \int_1 dx^- d^2 x_\perp T^{+-} + \frac{1}{2} \int_2 dx^+ d^2 x_\perp T^{--}, \tag{2.17}$$

where, from eq. (2.16),

$$T^{+-} = (\partial_- A^-)^2 + (\partial_i A^j)(\partial_i A^j) - (\partial_i A^j)(\partial_j A^i) \tag{2.18}$$

$$T^{--} = 4(\partial_+ A^i)^2 + (\partial_i A^-)^2 + 4(\partial_+ A^i)(\partial_i A^-) - 2\lambda A^-, \tag{2.19}$$

and the surfaces 1 and 2 are shown in Fig. 1. These expressions simplify somewhat when written in terms of φ, λ, and T^i. Inserting eqs. (2.8) and (2.10) into eq. (2.18), for example, and integrating by parts freely in the transverse directions, we find that the first term in eq. (2.17) can be written as

$$\frac{1}{2}\int_1 dx^- d^2x_\perp (\partial_i T^j)(\partial_i T^j). \tag{2.20}$$

Similarly, we may insert the relations

$$\partial_- A^- = -\partial_i T^i \tag{2.21}$$

and

$$A^- = -\frac{1}{\partial_-}\partial_i T^i - \frac{2}{\partial_\perp^2}(\lambda + \partial_+ \varphi) \tag{2.22}$$

into eq. (2.19), and use (2.5), to obtain

$$\frac{1}{2}\int_2 dx^+ d^2x_\perp T^{--} = 2\int_2 dx^+ d^2x_\perp \Big[(\partial_+ T^i)^2 + (\partial_+\varphi)\Big(\frac{1}{\partial_\perp^2}\lambda\Big)\Big]. \tag{2.23}$$

(We have again integrated by parts in the transverse directions repeatedly.) Now, the only subtle point concerns the first term on the RHS of eq. (2.23); we do not know the value of T^i on the boundary surfaces until the theory is solved. In the free theory this is no great difficulty—we can simply solve the theory and compute this term (it is zero[‡1])—but in the interacting case it is crucial that the generators be constructed only using the initial data. However, T^i must be integrable over the surface $x^+ = 0$. This requires at a minimum that T^i should fall to zero for $x^- \to \pm\infty$. Note that the field redefinition (2.8) insures that this is consistent; the equation of motion for T^i (eq. (2.11)) does not contain any terms that persist at $x^- = \pm\infty$, so that if $T^i = 0$ there initially, it will remain so under evolution in x^+. This should be contrasted with the equation for the full transverse gauge field A^i,

$$(4\partial_+\partial_- - \partial_\perp^2)A^i + \partial_i\varphi = 0. \tag{2.24}$$

Here it is *not* consistent to assume that A^i vanishes on the boundaries, unless $\varphi = 0$ for all x^+.

[‡1] Specifically, with the solution of the free theory in hand we can compute the integral over a finite-sized version of the surface shown in Fig. 1, and afterwards let the size become infinite. We find that the contributions involving T^i on the boundary surfaces vanish.

In a sense (which is necessarily somewhat imprecise in a continuum formulation), the field redefinition (2.10) is a separation of A^i into a "zero mode" part and a "normal mode" part, with the normal mode part assumed to vanish at large $|x^-|$. (Recall that T^i is to be smeared in test functions that vanish at $k^+=0$, so that it contains no x^--independent piece.) This is reminiscent of the situation in the discretized formulation [12,13,14,15], where the zero mode part of any bosonic field is a constrained function of the other fields in the theory, in this case φ. Examination of the equation of motion (2.24) shows that A^i must contain an x^--independent piece equal to $\frac{\partial_i}{\partial_\perp^2}\varphi$, or this equation cannot be satisfied.

Thus we conclude that T^i may be set to zero on the boundary surfaces, giving the following expression for P^-:

$$P^- = \frac{1}{2}\int_1 dx^- d^2x_\perp (\partial_i T^j)(\partial_i T^j) + 2\int_2 dx^+ d^2x_\perp (\partial_+\varphi)\Big(\frac{1}{\partial_\perp^2}\lambda\Big). \tag{2.25}$$

In a similar way we derive

$$P^+ = 2\int_1 dx^- d^2x_\perp (\partial_- T^i)^2 + \frac{1}{2}\int_2 dx^+ d^2x_\perp (\partial_i T^j)(\partial_i T^j). \tag{2.26}$$

Again, the term involving T^i on the boundary is zero, so that

$$P^+ = 2\int_1 dx^- d^2x_\perp (\partial_- T^i)^2. \tag{2.27}$$

It is now a simple matter to deduce the field algebra, by demanding that the Heisenberg relations reduce to the Euler-Lagrange equations. (In general it may also be necessary to demand that the algebra of generators be correctly realized, in order to fix all of the commutators uniquely.) The results are

$$[T^i(x^-,x_\perp), \partial^+ T^j(y^-,y_\perp)]_{x^+=0} = i\delta^{ij}\delta^{(3)}(\underline{x}-\underline{y}) \tag{2.28}$$

$$[\varphi(x^+,x_\perp), \lambda(y^+,y_\perp)] = i\delta(x^+-y^+)\partial_\perp^2\delta^{(2)}(x_\perp - y_\perp) \tag{2.29}$$

$$[A^i,\varphi] = [A^i,\lambda] = [\varphi,\varphi] = [\lambda,\lambda] = 0. \tag{2.30}$$

Thus the fields φ and $\frac{1}{\partial_\perp^2}\lambda$ are canonically conjugate with respect to x^--evolution.

Now let us solve the theory and compute the propagator. We begin by Fourier expanding the fields on their respective initial-value surfaces:

$$T^i(x^-,x_\perp)\Big|_{x^+=0} = \frac{1}{(2\pi)^{3/2}}\int \frac{dk^+ d^2k_\perp}{\sqrt{2k^+}}\Big(a_{\underline{k}i}e^{-i\underline{k}\cdot\underline{x}} + a^\dagger_{\underline{k}i}e^{i\underline{k}\cdot\underline{x}}\Big) \tag{2.31}$$

$$\varphi(x^+,x_\perp) = \frac{-i}{(2\pi)^{3/2}} \int \frac{dp^- d^2p_\perp}{\sqrt{2}} p_\perp^{1/2} \Big(f_{\overline{p}} e^{-\frac{i}{2}p^- x^+ + ip_\perp x_\perp} - f_{\overline{p}}^\dagger e^{\frac{i}{2}p^- x^+ - ip_\perp x_\perp} \Big) \tag{2.32}$$

$$\lambda(x^+,x_\perp) = \frac{1}{(2\pi)^{3/2}} \int \frac{dp^- d^2p_\perp}{\sqrt{2}} p_\perp^{3/2} \Big(g_{\overline{p}}\, e^{-\frac{i}{2}p^- x^+ + ip_\perp x_\perp} + g_{\overline{p}}^\dagger\, e^{\frac{i}{2}p^- x^+ - ip_\perp x_\perp} \Big), \tag{2.33}$$

where the factors of $p_\perp$ in (2.32) and (2.33) have been introduced for later convenience.[‡2] These last two expressions are already valid throughout all of spacetime, since φ and λ are independent of x^-. The solution for T^i is obtained by inserting appropriate x^+-dependent phases $e^{\pm\frac{i}{2}k^- x^+}$ with k^- given by the free-particle dispersion relation

$$k^- = \frac{k_\perp^2}{k^+}. \tag{2.34}$$

Thus

$$T^i(x^+,x^-,x_\perp) = \frac{1}{(2\pi)^{3/2}} \int \frac{dk^+ d^2k_\perp}{\sqrt{2k^+}} \Big(a_{\underline{k}i}\, e^{-\frac{i}{2}\frac{k_\perp^2}{k^+}x^+ - i\underline{k}\cdot\underline{x}} + a_{\underline{k}i}^\dagger e^{+\frac{i}{2}\frac{k_\perp^2}{k^+}x^+ + i\underline{k}\cdot\underline{x}} \Big). \tag{2.35}$$

The commutation relations (2.28)–(2.30) are realized by the Fock space relations

$$[a_{\underline{k}i}, a_{\underline{l}j}^\dagger] = \delta^{ij}\delta(k^+ - l^+)\delta^{(2)}(k_\perp - l_\perp) \tag{2.36}$$

$$[g_{\overline{p}}, f_{\overline{q}}^\dagger] = [f_{\overline{p}}, g_{\overline{q}}^\dagger] = \delta(p^+ - q^+)\delta^{(2)}(p_\perp - q_\perp) \tag{2.37}$$

and

$$[a,f] = [a,g] = [a^\dagger, f] = [a^\dagger, g] = [f, f^\dagger] = [g, g^\dagger] = 0 \tag{2.38}$$

If we wish, the mixed f and g commutators may be disentangled through the redefinitions

$$f_{\overline{p}} \equiv \frac{1}{\sqrt{2}}(b_{\overline{p}} + ic_{\overline{p}}) \qquad\qquad g_{\overline{p}} \equiv \frac{1}{\sqrt{2}}(b_{\overline{p}} - ic_{\overline{p}}), \tag{2.39}$$

where to reproduce (2.37)–(2.38) we must have

$$[a_{\overline{p}}, a_{\overline{q}}^\dagger] = -[b_{\overline{p}}, b_{\overline{q}}^\dagger] = \delta(p^- - q^-)\delta^{(2)}(p_\perp - q_\perp). \tag{2.40}$$

Thus we see clearly that the construction involves ghost states and an indefinite-metric Hilbert space. Unitarity and the full Poincaré invariance will be recovered by restricting to a suitable physical subspace, as discussed below.

‡2 We adopt the notation $\underline{k} \equiv (k^+, k_\perp)$, $\underline{x} \equiv (x^-, x_\perp)$, and $\underline{k}\cdot\underline{x} \equiv \frac{1}{2}k^+x^- - k_\perp x_\perp$. In addition, $\overline{p} = (p^-, p_\perp)$. In these and subsequent formulas, integrals over $k^\pm$ are understood to run from 0 to ∞, while integrals over transverse momenta run from $-\infty$ to ∞.

We now have all the ingredients in hand to compute the propagator. In fact, the operator solution we have obtained (eqs. (2.32), (2.33), and (2.35)) is identical to that obtained in ref. [5], so we may be confident that the propagator will have the ML form. For completeness, however, we shall here discuss the most singular component, $D^{--}(x)$. We have

$$D^{--}(x) = \vartheta(x^+)\langle 0|A^-(x)A^-(0)|0\rangle + \vartheta(-x^+)\langle 0|A^-(0)A^-(x)|0\rangle, \tag{2.41}$$

where $\vartheta(x) = 1$ for $x > 0$ and is zero otherwise. We construct the field A^- following eq. (2.22); it is the sum of a "transverse" part

$$A_T^-(x) = -\frac{\sqrt{2}}{(2\pi)^{3/2}} \sum_i \int dk^+ d^2k_\perp \frac{k^i}{(k^+)^{3/2}} \left(a_{\underline{k}i} e^{-ik\cdot x} + a^\dagger_{\underline{k}i} e^{ik\cdot x} \right) \tag{2.42}$$

and a "longitudinal" part

$$A_L^-(x) = \frac{\sqrt{2}}{(2\pi)^{3/2}} \int \frac{dp^- d^2p_\perp}{p_\perp^{1/2}} \Big[\Big(g_{\overline{p}} - \frac{p^-}{2p_\perp} f_{\overline{p}} \Big) e^{-\frac{i}{2}p^- x^+ + ip_\perp x_\perp} + \Big(g^\dagger_{\overline{p}} - \frac{p^-}{2p_\perp} f^\dagger_{\overline{p}} \Big) e^{+\frac{i}{2}p^- x^+ - ip_\perp x_\perp} \Big]. \tag{2.43}$$

From these, and the commutation relations (2.36)–(2.38), we easily compute, for example,

$$\langle 0|A^-(x)A^-(0)|0\rangle = \frac{2}{(2\pi)^3} \int dk^+ d^2k_\perp \frac{k_\perp^2}{(k^+)^3} e^{-ikx} - \frac{2}{(2\pi)^3} \int dk^- d^2k_\perp \frac{k^-}{k_\perp^2} e^{-\frac{i}{2}k^- x^+ + ik_\perp x_\perp}. \tag{2.44}$$

The expectation value of the fields in the other order is of course just the complex conjugate of (2.44). This is just the ML form of D^{--} (see for example ref. [5]), written in terms of light-cone variables.[‡3] The small-k^+ singularity in the first term in eq. (2.44) is canceled by the large-k^- region in the second term, as is easily seen by making the change of variables $k^- = \frac{k_\perp^2}{k^+}$ in the first term.

Finally, selection of the physical subspace follows exactly as discussed in ref. [5]. The physical subspace is defined by the requirement that Maxwell's equations and the Poincaré algebra are obtained in matrix elements between physical states. Now, the only Maxwell

[‡3] Note that the choice of test functions in which T^i is to be smeared insures that the change from ordinary spacetime variables to light-cone variables is well defined.

equation that is not satisfied at the operator level is (2.5). Therefore, physical states are those between which λ has vanishing matrix elements, or equivalently those satisfying

$$g_{\overline{p}}|\text{phys}\rangle = 0. \tag{2.45}$$

It is straightforward to show that states defined in this way have positive norm, so that unitarity holds in this subspace. Furthermore, one can check that the extra terms that occur in the algebra of the Poincaré generators are all proportional to λ, so that the correct Poincaré algebra is recovered in matrix elements between states satisfying (2.45).

3. Gauge Fields in Discretized Light-Cone Quantization

In this section we shall remark briefly on the relevance of the above when we require the gauge field to periodic in x^-.[‡4] This formalism is widely used in setting up actual numerical simulations of light-cone field theories *á la* Tamm and Dancoff [16]. It is also of interest simply as a regulator of the small-k^+ region of the theory.

Our first remark is by now well known: it is impossible to realize the light cone gauge $A^+ = 0$ when the gauge field is required to be periodic in x^-. The easiest way to see this is to note that under a gauge transformation

$$A^+ \to A^+ + \partial^+ \Lambda. \tag{3.1}$$

In order to preserve the periodicity of the other components of A^μ (or the boundary conditions imposed on some fermi field coupled to A_μ), the function Λ must itself be periodic. Thus any x^--independent part of A^+ is in fact gauge invariant, and so an arbitrary A^+ cannot be brought to zero by a gauge transformation. The best we can do is to set everything but the x^--independent part of A^+ to zero; this is equivalent to the gauge choice $\partial_- A^+ = 0$. The $k^+ = 0$ mode in A^+, which can have nontrivial dependence on the transverse coordinates, must be retained in the theory.

Another difference between the discretized and the continuum theories is that in the discretized case it is natural, due to the explicit imposition of boundary conditions on A_μ, to eliminate all of the residual gauge freedom. The condition $\partial_- A^+ = 0$ is preserved by x^--independent gauge transformations, so that one is free to impose further gauge

[‡4] We shall not consider the case where A_μ is taken to be antiperiodic on x^-, as the fermion bilinears we eventually wish to couple to A_μ are necessarily periodic.

conditions that set various x^--independent modes of A^- and A^i to zero. Kalloniatis and Pauli have proposed one complete gauge fixing for discretized electrodynamics [17]. In their construction the residual gauge freedom has been used to eliminate a maximal number of components of the gauge field, so that all that remain are physical fields, and various constrained zero modes. These latter objects occur generically in discretized theories involving bosonic fields; they satisfy constraint relations, and must be solved for in terms of the dynamical fields of the theory [12,13,14,15].

The result is a theory which is regulated at small k^+, and which involves only physical degrees of freedom. As noted previously, in the continuum theory we *could* fix boundary conditions at $x^- = \pm\infty$, thereby eliminating the residual gauge freedom. The resulting theory would not be isomorphic to that described in ref. [5], of course, but would be a candidate formulation of light-cone gauge field theory.[‡5] (The two formulations would be related by a gauge transformation that rotates away the unphysical fields.) All such formulations that have so far been considered, however, give inconsistent results in perturbative calculations, even for gauge-invariant quantities. In particular, we typically obtain the principal value prescription for the spurious pole.

One lesson we take from this is that even though the occurrence of the spurious pole is related to an unfixed gauge freedom, it matters what we do to regulate it. It is not true that it is simply a "gauge artifact," and any way of making it finite is acceptable. This leads us to ask whether anything analogous occurs in the discretized formulation. That is, does using discretization as a regulator result in some inconsistency, analogous to using a principal value in the continuum? In the latter case, certain difficulties are apparent already at the level of the free theory: we have positive (negative) energy quanta propagating into the backward (forward) light cone. It will be important to check the viability of the discretized theory in perturbative calculations, once a fully satisfactory formulation is achieved. This is a challenge because the relations that determine the constrained zero modes are difficult to solve. Work in this direction has been recently reported by Kalloniatis and Pauli [17,18].

One aspect of the discretized theory does have an analog in the continuum theory described here. Those modes of A^i that are constant in both x^- and $x_\perp$ are the only ones that are not either obviously dynamical (*i.e.*, there is no classical equation of motion for them) or determined by a constraint relation. They do, however, appear in *e.g.*, the

[‡5] Note that for the light-cone gauge one cannot eliminate the residual gauge freedom when quantizing on a spacelike surface.

constraint relation for the constrained part of the fermi field. Physically, these degrees of freedom correspond to photons with k^+ and $k_\perp$ vanishing, that is, quanta propagating precisely along the surface $x^+ = 0$. In the continuum we can discard these by a suitable choice of test functions, and feel reasonably confident that no serious damage is done because the subset of the theory that is excluded is terrifically small. Here, however, they are two of a countably infinite set of degrees of freedom, and so may need to be included. The point is that, precisely because the quanta they represent propagate along $x^+ = 0$, they should be included in the theory as fields initialized along a surface of constant x^-, in the same way as φ and λ. Work on including these is currently in progress [19].

4. Discussion

We have seen that the results of ref. [5] can be reproduced in light-cone quantization if proper attention is paid to the degrees of freedom and where they must be initialized. In particular, the unphysical fields φ and λ must be initialized on a surface of equal x^-, and satisfy equal-x^- canonical commutation relations. A further complication is the necessity of including boundary contributions in the calculation of conserved charges. This becomes especially tedious in the interacting theory; the fully interacting case, with application to light-cone bound state equations, will be discussed elsewhere [20].

In this language, it is the appearance of the second characteristic surface which brings in the "dual" gauge vector $n^*_\mu = (1,0,0,-1)$ needed to formulate the ML prescription [2]. It is also clear why the ML prescription is impossible to realize in the context of strictly x^+-ordered (light-cone) perturbation theory—we simply do not have a formalism in which all fields are evolved from a surface of constant x^+. This is always a possibility when quantizing on characteristic surfaces, though in certain cases it may be avoided. The discretized formulation is not precisely analogous, in part because we cannot realize the light-cone gauge for periodic gauge fields. It does appear to have some similar features, however, in particular the need to include fields that live along other characteristics. It will be interesting to investigate whether discretized light-cone QED suffers from pathologies analogous to those that seem to plague continuum formulations in the light cone gauge, when the gauge is completely fixed.

Finally, we should perhaps emphasize that this entire discussion has had as its motivation the successful formulation of perturbation theory for gauge theories. The relevance of the ML prescription for the kind of *nonperturbative* calculations everyone would like to do is somewhat unclear. Perhaps these problems only arise if we insist on setting up a calculational scheme in terms of unphysical, gauge-dependent quantities, like propagators and vertices. It may very well be that we have to be *more* careful in this case than we would have to be if we we formulated and solved the theory in a physical way from the start.

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