10-15-1994

On the Discretized Light Cone Quantization of Electrodynamics

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(Received 27 May 1994)

Discretized light-cone quantization of (3+1)-dimensional electrodynamics is discussed, with careful attention paid to the interplay between gauge choice and boundary conditions. In the zero longitudinal momentum sector of the theory of general gauge fixing is performed, and the corresponding relations that determine the zero modes of the gauge field are obtained. One particularly natural gauge choice in the zero mode sector is identified, for which the constraint relations are simplest and the fields may be taken to satisfy the usual canonical commutation relations. The constraints are solved in perturbation theory, and the Poincaré generators P^μ are constructed. The effect of the zero mode contributions on the one-loop fermion self-energy is studied.

PACS number(s): 11.10.Ef, 11.10.Gh, 12.20.Ds

I. INTRODUCTION

Light-cone quantization, or more properly quantization on a null plane [1], seems to offer several advantages over the more traditional equal-time quantization for a nonperturbative treatment of field theories. There are arguments, for example, that certain Lorentz boosts are in the kinematical subgroup and that the vacuum structure is simpler. There has recently been considerable effort devoted to exploiting these advantages in the context of a Tamm-Dancoff-style solution of field theory [2,3]. For an overview of this work with many references, see Ref. [4]. This approach has been strikingly successful in two spacetime dimensions [5–12], and encouraging results have also been obtained recently in four-dimensional models [13–16]. For a complete attack on four-dimensional QCD to be feasible, however, many technical obstacles remain to be overcome.

A particularly simple framework for actual calculations is that of "discretized" light-cone quantization (DLCQ), in which the theory is defined on a light-cone "torus" [5]. It then possesses a discrete momentum-space basis, which regulates infrared divergences and is ready-made for numerical analysis. The goal of this approach is to give a controllable formulation of a quantum field theory in terms of a Hamiltonian eigenvalue equation and then to solve it, in general, numerically.

Certain formal aspects of this approach, however, are not yet completely under control. One such area concerns the zero modes (in the Fourier sense) of bosonic fields. To illustrate the basic problem, let us consider a self-interacting scalar field in 1+1 dimensions, for which the Euler-Lagrange equation is

$$(4\partial_-\partial_+ + m^2)\phi = -\lambda\phi^3. \tag{1.1}$$

(Our notational conventions are summarized in Appendix A.) After imposing periodic boundary conditions on the finite interval $-L \leq x^- \leq L$, we can expand the field in discrete Fourier modes with momenta $k^+ = \frac{n\pi}{L}$, n an even integer. We then project out the zero mode ($n = 0$) by integrating both sides of the equation over the entire interval, obtaining

$$m^2\overset{\circ}{\phi} = -\lambda\overset{\circ}{\phi}{}^3 - \frac{\lambda}{2L}\int_{-L}^{L} dx^-(3\overset{\circ}{\phi}\varphi^2 + \varphi^3). \tag{1.2}$$

Here $\overset{\circ}{\phi}$ is the zero mode and φ is the complementary "normal mode" part: $\varphi \equiv \phi - \overset{\circ}{\phi}$. The important thing to note is that the time (x^+) derivative has dropped out due to the chosen boundary conditions. The zero mode is therefore a *constrained* field for which we cannot specify independent quantum commutation relations [17–20]. Furthermore, $\overset{\circ}{\phi}$ is needed for the computation of, e.g., the Poincaré generators. The nonlinear operator constraint (1.2) must therefore be solved before the Hamiltonian can even be written down.

There is a striking simplification that occurs elsewhere in the theory, however: The Fock vacuum is an exact eigenstate of the full Hamiltonian. This follows from light-cone momentum (P^+) conservation and the observation that the zero mode does not correspond to a degree of freedom—that is, there is no $P^+ = 0$ quantum in the theory. The bare vacuum is thus the only state in the theory with $P^+ = 0$ and must therefore be an exact eigenstate of the full Hamiltonian. This is a highly desirable feature if we wish to have a constituent picture of relativistic bound states and describe, for example, a baryon as primarily a three-quark state plus a few higher Fock states in the manner of Tamm and Dancoff. In equal-time quantization, where the physical vacuum state is an infinite superposition of states with arbitrarily large numbers of bare quanta, it would be extremely difficult to describe a baryon in this fashion. In this case a sensible constituent description would be in terms of "quasiparticles," perhaps corresponding loosely to the quarks of the constituent quark model. These would be complicated collective excitations above the physical vacuum state. The difficulty here, of course, is that without knowledge of the full solution of the theory we have no idea how to connect these quasiparticle states to the bare states (in terms of which the Hamiltonian is easily formulated).

In DLCQ the problem of the vacuum is apparently shifted to that of obtaining solutions to the constraint equations for the zero modes. Some preliminary support for this view is provided by considering the model of Eq. (1.1) with $m^2 < 0$, in which case we anticipate spontaneous breakdown of the reflection symmetry $\phi \to -\phi$. Here we find [21–23] that there are multiple solutions to the constraint (1.2), and we must choose one to use in formulating the theory. This choice is analogous to what in conventional language we would call the choice of vacuum state. The various solutions contain c-number pieces which produce the possible vacuum expectation values of ϕ. The properties of the strong-coupling phase transition in this model are also reproduced, including its second-order nature and a reasonable value for the critical coupling [23,24]. (For an earlier study of this phase transition in DLCQ without the zero mode, see [25].) So in some cases, at least, physics which we normally associate with the vacuum can be manifested in these zero modes, in a formalism where the vacuum state itself is simple.

We should perhaps emphasize that, apart from the question of whether or not vacuum expectation values arise, solving the constraint equations really amounts to determining the Hamiltonian (and other Poincaré generators). In general, P^- becomes very complicated when the zero mode contributions are included; this is in some sense the price we pay to achieve a formulation with a simple vacuum.[1] The other Poincaré generators apparently also receive contributions from the zero modes, and it becomes important to check whether the supposedly kinematical ones, P^+ and P^i, remain kinematical when the zero modes are included. As we shall see, this is not always guaranteed to be the case.

When considering a gauge theory, there is another "zero mode" problem associated with the choice of gauge in the compactified case. This subtlety, however, is not particular to the light cone; indeed, its occurrence is quite familiar in equal-time quantization on a torus [28]. In the present context, the difficulty is that the zero mode in A^+ is in fact gauge invariant, so that the light-cone gauge $A^+ = 0$ cannot be reached. Thus we have a pair of interconnected problems: first, a practical choice of gauge and, second, the presence of constrained zero modes of the gauge field. In two recent papers [29,30], these problems were separated and consistent gauge-fixing conditions were introduced to allow isolation of the dynamical and constrained fields. In the present paper we shall generalize the gauge fixing described in Ref. [30] and construct the Poincaré generators in perturbation theory. Our aim is to study the types of operators induced by the zero modes and to examine their effects on the perturbative renormalization of this theory.

We begin in the next section by reviewing the approach laid out in Ref. [30] and separating the constrained zero modes from the dynamical degrees of freedom within a general gauge fixing. We obtain the relations that define the constrained operators and show that there is a unique choice of gauge in the zero mode sector for which the naively kinematical operators P^i retain their free-field forms, so that the usual (free-field) commutation relations can be employed. In Sec. III we solve the constraints in perturbation theory and construct the Hamiltonian to lowest nontrivial order. In Sec. IV we study the effects of the zero mode contributions on the one-loop fermion self-energy. Some discussion and our conclusions are presented in Sec. V.

II. GAUGE FIXING AND THE ZERO-MOMENTUM MODES

With the standard Lagrangian density for electrodynamics,

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{2.1}$$

the equations of motion are the familiar Dirac equation

$$(i\not{D} - m)\psi = 0 \tag{2.2}$$

and Maxwell's equation

$$\partial_\mu F^{\mu\nu} = gJ^\nu, \tag{2.3}$$

where $D_\mu = \partial_\mu + igA_\mu$ and $J^\mu \equiv :\bar{\psi}\gamma^\mu\psi:$. Written out explicitly in terms of the various gauge field components and the spinor projections defined in Appendix A, these become

$$(2i\partial_+ - gA^-)\psi_+ = [-i\alpha^i\partial_i + m\beta + g\alpha^i A_i]\psi_-, \tag{2.4}$$

$$(2i\partial_- - gA^+)\psi_- = [-i\alpha^i\partial_i + m\beta + g\alpha^i A_i]\psi_+, \tag{2.5}$$

$$2\partial_+\partial_- A^+ - 2(\partial_-)^2 A^- - 2\partial_-\partial_j A^j - \partial_\perp^2 A^+ = gJ^+, \tag{2.6}$$

$$2\partial_+\partial_- A^- - 2(\partial_+)^2 A^+ - 2\partial_+\partial_j A^j - \partial_\perp^2 A^- = gJ^-, \tag{2.7}$$

$$(4\partial_+\partial_- - \partial_\perp^2)A^i + \partial_i\partial_+ A^+ + \partial_i\partial_- A^- + \partial_i\partial_j A^j = gJ^i. \tag{2.8}$$

[1] It may be possible to think of the discretization as a cutoff which removes states with $0 < p^+ < \pi/L$ and the zero mode contributions to the Hamiltonian as effective interactions that restore the discarded physics. We shall not pursue this idea in detail here, except to note that from the light-cone power-counting analysis of Wilson [26,27] it is clear that there will be a huge number of allowed operators.

Observe that in the traditional treatment, choosing the light-cone gauge $A^+ = 0$ enables Eq. (2.6) to be solved for A^-. In any case the spinor projection ψ_- is constrained and determined by Eq. (2.5).

Discretization is achieved by putting the theory in a light-cone "box," with $-L_\perp \leq x^i \leq L_\perp$ and $-L \leq x^- \leq L$, and imposing boundary conditions on the fields. The choices of boundary conditions are constrained by the need to be consistent with the equations of motion. Because the gauge field couples to a fermion bilinear, which is necessarily periodic in all coordinates, A_μ must be taken to be periodic in both x^- and $x_\perp$. We have more flexibility with the Fermi field, and it is most convenient to choose this to be periodic in $x_\perp$ and antiperiodic in x^-. This eliminates the zero longitudinal momentum mode while still allowing an expansion of the field in a complete set of basis functions.

The functions used to expand the fields may be taken to be plane waves, and for periodic fields these will of course include zero-momentum modes. Let us define, for a periodic quantity f, its longitudinal zero mode

$$\langle f \rangle_0 \equiv \frac{1}{2L} \int_{-L}^{L} dx^- f(x^-, x_\perp) \tag{2.9}$$

and the corresponding normal mode part

$$f_n \equiv f - \langle f \rangle_0. \tag{2.10}$$

We shall further denote the "global zero mode"—the mode independent of all the spatial coordinates—by $\langle f \rangle$:

$$\langle f \rangle \equiv \frac{1}{\Omega} \int_{-L}^{L} dx^- \int_{-L_\perp}^{L_\perp} d^2x_\perp f(x^-, x_\perp). \tag{2.11}$$

Finally, the quantity which will be of most interest to us is the "proper zero mode," defined by

$$\overset{\circ}{f} \equiv \langle f \rangle_0 - \langle f \rangle. \tag{2.12}$$

By integrating over the appropriate direction(s) of space, we can project the equations of motion onto the various sectors. Previous work on the formulation of QED in DLCQ [13,31] has been implicitly carried out in the normal mode sector, and many of these results may be carried over without modification. The global zero mode sector requires some special treatment and, in fact, turns out to be irrelevant for the perturbative calculations we shall present here. A brief description of its features and a proof that it can be ignored to lowest nontrivial order in perturbation theory are given in Appendix B.

Thus we concentrate our attention on the proper zero mode sector, in which the equations of motion become

$$-\partial_\perp^2 \overset{\circ}{A}{}^+ = g\overset{\circ}{J}{}^+, \tag{2.13}$$

$$-2(\partial_+)^2 \overset{\circ}{A}{}^+ - \partial_\perp^2 \overset{\circ}{A}{}^- - 2\partial_i \partial_+ \overset{\circ}{A}{}^i = g\overset{\circ}{J}{}^-, \tag{2.14}$$

$$-\partial_\perp^2 \overset{\circ}{A}{}^i + \partial_i \partial_+ \overset{\circ}{A}{}^+ + \partial_i \partial_j \overset{\circ}{A}{}^j = g\overset{\circ}{J}{}^i. \tag{2.15}$$

We first observe that Eq. (2.13), the projection of Gauss' law, is a constraint which determines the proper zero mode of A^+ in terms of the current J^+:

$$\overset{\circ}{A}{}^+ = g\frac{1}{\partial_\perp^2}\overset{\circ}{J}{}^+. \tag{2.16}$$

[Note that the integral operator $(\partial_\perp^2)^{-1}$ is well defined in this sector [30].] Equations (2.14) and (2.15) then determine the zero modes $\overset{\circ}{A}{}^-$ and $\overset{\circ}{A}{}^i$.

Equation (2.16) is clearly incompatible with the strict light-cone gauge $A^+ = 0$, which is most natural in light-cone analyses of gauge theories. Here we encounter a common problem in treating axial gauges on compact spaces [28], which has nothing to do with light-cone quantization *per se*. The point is that any x^--independent part of A^+ is in fact gauge invariant, since, under a gauge transformation,

$$A^+ \to A^+ + 2\partial_-\Lambda, \tag{2.17}$$

where Λ is a function periodic in all coordinates.[2] Thus it is not possible to bring an arbitrary gauge field configuration to one satisfying $A^+ = 0$ via a gauge transformation, and the light-cone gauge is incompatible with the chosen boundary conditions. The closest we can come is to set the normal mode part of A^+ to zero, which is equivalent to

$$\partial_- A^+ = 0. \tag{2.18}$$

This condition does not, however, completely fix the gauge—we are free to make arbitrary x^--independent gauge transformations without undoing Eq. (2.18). We may therefore impose further conditions on A_μ in the zero mode sector of the theory.

To see what might be useful in this regard, let us consider solving Eq. (2.15). We begin by acting on Eq. (2.15) with ∂_i. The transverse field $\overset{\circ}{A}{}^i$ then drops out, and we obtain an expression for the time derivative of $\overset{\circ}{A}{}^+$:

$$\partial_+ \overset{\circ}{A}{}^+ = g\frac{1}{\partial_\perp^2}\partial_i \overset{\circ}{J}{}^i. \tag{2.19}$$

[This can also be obtained by taking a time derivative of Eq. (2.16) and using current conservation to reexpress the right-hand side in terms of J^i.] Inserting this back into Eq. (2.15), we then find, after some rearrangement,

$$-\partial_\perp^2 \left(\delta_j^i - \frac{\partial_i\partial_j}{\partial_\perp^2}\right)\overset{\circ}{A}{}^j = g\left(\delta_j^i - \frac{\partial_i\partial_j}{\partial_\perp^2}\right)\overset{\circ}{J}{}^j. \tag{2.20}$$

Now the operator $(\delta_j^i - \partial_i\partial_j/\partial_\perp^2)$ is nothing more than the projector of the two-dimensional transverse part of the vector fields $\overset{\circ}{A}{}^i$ and $\overset{\circ}{J}{}^i$. No trace remains of the longitudinal projection of the field $(\partial_i\partial_j/\partial_\perp^2)\overset{\circ}{A}{}^j$ in Eq. (2.20).

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[2]The gauge transformation must also preserve the boundary conditions on the other fields; thus, e.g., $\Lambda \sim x^-$ is not in general allowed. See, however, Appendix B.

This reflects precisely the residual gauge freedom with respect to x^--independent transformations. To determine the longitudinal part, an additional condition is required.

More concretely, the general solution to Eq. (2.20) is

$$\overset{\circ}{A}{}^i = -g\frac{1}{\partial_\perp^2}\overset{\circ}{J}{}^i + \partial_i\varphi(x^+, x_\perp), \tag{2.21}$$

where φ must be independent of x^-, but is otherwise arbitrary. Imposing a condition on, say, $\partial_i \overset{\circ}{A}{}^i$ will uniquely determine φ.[3] In Ref. [30], for example, the condition

$$\partial_i \overset{\circ}{A}{}^i = 0 \tag{2.22}$$

was proposed as being particularly natural. This choice, taken with the other gauge conditions we have imposed, has been called the "compactification gauge." In this case,

$$\varphi = g\frac{1}{(\partial_\perp^2)^2}\partial_i \overset{\circ}{J}{}^i. \tag{2.23}$$

Of course, other choices are also possible. For example, we might generalize Eq. (2.23) to

$$\varphi = \alpha g\frac{1}{(\partial_\perp^2)^2}\partial_i \overset{\circ}{J}{}^i, \tag{2.24}$$

with α a real parameter. The gauge condition corresponding to this solution is

$$\partial_i \overset{\circ}{A}{}^i = -g(1-\alpha)\frac{1}{\partial_\perp^2}\partial_i \overset{\circ}{J}{}^i. \tag{2.25}$$

We shall refer to this as the "generalized compactification gauge." An arbitrary gauge field configuration B^μ can be brought to one satisfying Eq. (2.25) via the gauge function

$$\Lambda(x_\perp) = -\frac{1}{\partial_\perp^2}\left[g(1-\alpha)\frac{1}{\partial_\perp^2}\partial_i \overset{\circ}{J}{}^i + \partial_i \overset{\circ}{B}{}^i\right]. \tag{2.26}$$

This is somewhat unusual in that $\Lambda(x_\perp)$ involves the sources as well as the initial field configuration, but this is perfectly acceptable. More generally, φ can be any (dimensionless) function of gauge invariants constructed from the fields in the theory, including the currents $J^\pm$. For our purposes Eq. (2.25) suffices.

We now have relations defining the proper zero modes of A^i:

$$\overset{\circ}{A}{}^i = -g\frac{1}{\partial_\perp^2}\left(\delta^i_j - \alpha\frac{\partial_i\partial_j}{\partial_\perp^2}\right)\overset{\circ}{J}{}^j, \tag{2.27}$$

as well as $\overset{\circ}{A}{}^+$ [Eq. (2.16)]. All that remains is to use the final constraint [Eq. (2.14)] to determine $\overset{\circ}{A}{}^-$. Using Eqs. (2.19) and (2.25), we find that Eq. (2.14) can be written as

$$\partial_\perp^2 \overset{\circ}{A}{}^- = -g\overset{\circ}{J}{}^- - 2\alpha g\frac{1}{\partial_\perp^2}\partial_+\partial_i \overset{\circ}{J}{}^i. \tag{2.28}$$

After using the equations of motion to express $\partial_+\overset{\circ}{J}{}^i$ in terms of the dynamical fields at $x^+ = 0$, this may be straightforwardly solved for $\overset{\circ}{A}{}^-$ by inverting the $\partial_\perp^2$. In what follows, however, we shall have no need of $\overset{\circ}{A}{}^-$. It does not enter the Hamiltonian, for example; as usual, it plays the role of a multiplier to Gauss' law [Eq. (2.15)], which we are able to implement as an operator identity.

Now, since different choices for φ merely correspond to different gauge choices in the zero mode sector, we expect that physical quantities should be independent of the specific φ we choose [i.e., for the family of solutions defined by Eq. (2.24) physical quantities should be independent of the parameter α]. It may be, however, that particular choices for φ lead to particularly simple formulations. It is instructive in this regard to examine the naively kinematical Poincaré generators P^+ and P^i, and check whether they remain kinematical when the zero mode contributions are included.

The operators P^μ are defined by

$$P^\mu = \tfrac{1}{2}\int dx^- d^2x_\perp T^{+\mu}, \tag{2.29}$$

where we take for the energy-momentum tensor the gauge-invariant form

$$T^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda - gJ^\mu A^\nu + i\overline{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L}, \tag{2.30}$$

with $\mathcal{L}$ given in Eq. (2.1). Equation (2.30) differs by the addition of a total divergence from what we obtain by a straightforward application of Noether's theorem. For P^+ the relevant component is

$$T^{++} = 4(\partial_- A^i_n)^2 + 4(\partial_- A^i_n)(\partial_i \overset{\circ}{A}{}^+) + (\partial_i \overset{\circ}{A}{}^+)^2 - gJ^+\overset{\circ}{A}{}^+ + 4i\psi_+^\dagger\partial_-\psi_+. \tag{2.31}$$

The first and last terms in (2.31) just give the usual normal mode contribution to P^+. The second term vanishes upon integration in x^-. Finally, the remaining two terms combine, after a transverse integration by parts, to give a contribution to P^+:

$$\tfrac{1}{2}\int dx^- d^2x_\perp \overset{\circ}{A}{}^+(-\partial_\perp^2 \overset{\circ}{A}{}^+ - g\overset{\circ}{J}{}^+). \tag{2.32}$$

This vanishes upon implementing the constraint (2.13). Thus P^+ remains kinematical, even with the zero modes present.

For the P^i we require

$$T^{+i} = -(\partial_+\overset{\circ}{A}{}^+)(\partial_i\overset{\circ}{A}{}^+) - (\partial_j\overset{\circ}{A}{}^+)(\partial_i\overset{\circ}{A}{}^j) + (\partial_j\overset{\circ}{A}{}^+)(\partial_j\overset{\circ}{A}{}^i) - g\overset{\circ}{J}{}^+\overset{\circ}{A}{}^i + \cdots, \tag{2.33}$$

where we have omitted the purely normal mode contri-

[3]We could also refuse to completely fix the gauge and treat φ as a degree of freedom. It would be unphysical, however, and would have to be removed by restricting to a suitable physical subspace. See Ref. [32] for an example of this type of construction in a continuum formulation.

bution and terms that vanish upon integration. The last two terms cancel upon integration by parts and application of the constraint (2.13). The first two combine to give a contribution to P^i:

$$-\tfrac{1}{2}\int dx^- d^2x_\perp (\partial_i \overset{\circ}{A}{}^+)(\partial_+ \overset{\circ}{A}{}^+ + \partial_j \overset{\circ}{A}{}^j). \tag{2.34}$$

Clearly, P^i will contain zero mode contributions, and hence will be "interacting," unless

$$\partial_+ \overset{\circ}{A}{}^+ + \partial_j \overset{\circ}{A}{}^j = 0. \tag{2.35}$$

This corresponds to taking $\alpha = 0$ in (2.27). Interestingly, this condition amounts to a zero mode projection of the *Lorentz* gauge condition

$$\partial_\mu \overset{\circ}{A}{}^\mu = 0. \tag{2.36}$$

What does it mean for P^i to not have the same form as in free-field theory? In this case it will be impossible to realize the Heisenberg equation

$$[\psi_+, P^i] = -i\partial_i \psi_+ \tag{2.37}$$

with a *simple* anticommutation relation between ψ_+ and $\psi_+^\dagger$. In order to obtain Eq. (2.37) through $\mathcal{O}(g^2)$, we would have to take

$$\{\psi_+(\underline{x}), \psi_+^\dagger(\underline{x}')\} = \Lambda_+[\delta^{(3)}(\underline{x} - \underline{x}') + \mathcal{O}(g^2\alpha) + \cdots], \tag{2.38}$$

with the $\mathcal{O}(g^2\alpha)$ term in (2.38) chosen so that the part of $[\psi_+, P^i]$ coming from the interacting terms in P^i is canceled by contribution from the free-field part of P^i and the $\mathcal{O}(g^2\alpha)$ piece of the anticommutator.

In fact, we could determine the required anticommutation relation as follows. Consider the theory with $\alpha = 0$, where since the P^i have their usual free-field forms the standard canonical anticommutator for ψ_+ is the correct one. Now perform a redefinition of ψ_+ that corresponds to the gauge transformation that would take us from $\alpha = 0$ to $\alpha \neq 0$, specifically

$$\psi_+ = e^{-ig\Lambda}\psi'_+, \tag{2.39}$$

with

$$\Lambda = \alpha g \frac{1}{(\partial_\perp^2)^2}\partial_j \overset{\circ}{J}{}^j. \tag{2.40}$$

It is straightforward to check that when written in terms of ψ'_+, the P^μ have the same forms they would have if we had started with $\alpha \neq 0$ in Eq. (2.27). In particular, the P^i acquire a term equal to (2.34).[4] Thus ψ'_+ satisfies the commutation relations necessary to obtain Eq. (2.37) for $\alpha \neq 0$, and we can simply compute these by inverting the redefinition (2.39) and using the known commutation relations for ψ_+. All of this is not really necessary, however. The point is that different values of α are physically equivalent; they are just related by redefinitions of ψ_+. What is special about $\alpha = 0$ [or, more generally, $\varphi = 0$ in Eq. (2.21)] is that this is the unique choice for which simple commutation relations among the fields are possible. Thus it is most sensible to take $\alpha = 0$ from the very beginning, and for the remainder of the paper this is what we shall do.

It is perhaps also worth noting that $\alpha = 0$ results in the simplest constraint relation for $\overset{\circ}{A}{}^-$ [see Eq. (2.28) and the discussion following Eq. (3.1)]. Indeed, in this case all of the constrained zero modes satisfy

$$\overset{\circ}{A}{}^\mu = -g\frac{1}{\partial_\perp^2}\overset{\circ}{J}{}^\mu, \tag{2.41}$$

which has a pleasing symmetry.

Our next task is to solve the constraint relations for the determined fields and construct the dynamical operators. As a prelude to the next section, let us briefly remark that because the transverse currents themselves depend on $\overset{\circ}{A}{}^i$, the structure of Eq. (2.27) is somewhat more complicated than a first glance reveals. Nonperturbative solutions to these constraints have so far proved difficult to obtain. Nevertheless, an important first step toward understanding the implications of the zero modes is to examine them in perturbation theory. Thus we shall now pursue a perturbative solution for the $\overset{\circ}{A}{}^i$. This is equivalent to a Fredholm iterative treatment [30].

III. PERTURBATIVE FORMULATION

We now wish to solve the constraint relations (2.16), (2.27), and (2.28) for the zero modes and compute the dynamical operators of the theory. The components of J^μ are given in terms of $\psi_\pm$ by

$$J^\pm = 2\psi_\pm^\dagger \psi_\pm, \tag{3.1a}$$

$$J^i = \psi_+^\dagger \alpha^i \psi_- + \psi_-^\dagger \alpha^i \psi_+. \tag{3.1b}$$

From these and an inspection of the constraint equation (2.5) for ψ_-, we can easily identify which zero modes are simple and which are difficult to compute. The field $\overset{\circ}{A}{}^+$ is trivially obtained from Eq. (2.16), since it depends only on the dynamical part of the Fermi field ψ_+. The transverse fields $\overset{\circ}{A}{}^i$ are more complicated, since $\overset{\circ}{J}{}^i$ depends on both $\overset{\circ}{A}{}^i$ and $\overset{\circ}{A}{}^+$. Thus Eq. (2.27) actually determines $\overset{\circ}{A}{}^i$ implicitly, and obtaining a general solution for $\overset{\circ}{A}{}^i$ is quite difficult in the quantum theory. In some sense this is to be expected, however: The complexity normally associated with the vacuum state when quantizing on a spacelike surface has to go somewhere. Finally, $\overset{\circ}{A}{}^-$ will be as difficult to determine as the $\overset{\circ}{A}{}^i$ unless $\alpha = 0$; using Eq. (2.4) to express the x^+ derivative

[4]We are being somewhat cavalier here about issues of operator ordering, etc., which affect the precise form of the field redefinition (2.39).

of J^i in terms of the fields on $x^+ = 0$ will introduce $\mathring{A}^-$ into the right-hand side of Eq. (2.28). If $\alpha = 0$, however, then Eq. (2.28) allows a straightforward computation of $\mathring{A}^-$ in terms of ψ_+, A_n^i, and $\mathring{A}^i$. In this case the only approximations necessary to calculate $\mathring{A}^-$ are those needed for the computation of $\mathring{A}^i$.

We shall now construct a perturbative solution to Eq. (2.27) and study the structure of the theory to lowest nontrivial order. This requires constructing the Hamiltonian through terms of $\mathcal{O}(g^2)$, which in turn corresponds to taking the $\mathcal{O}(g)$ solution for $\mathring{A}^i$. We obtain this simply by setting $g = 0$ in the current $\mathring{J}^i$ or, in other words, by using the zeroth-order expression for ψ_-:

$$\psi_-^{(0)} = \frac{1}{2i\partial_-}(-i\alpha^i\partial_i + m\beta)\psi_+. \tag{3.2}$$

Inserting the Fourier expansion of the field ψ_+, we then find that the (normal-ordered) proper zero mode of the transverse current is given to this order by

$$:\mathring{J}^i: = -\frac{2}{\Omega}\sum_{s,\underline{k},\underline{k}'}{}' \frac{\delta_{k^+,k'^+}}{k^+}\left[\epsilon_{2s}^i(\epsilon_{-2s}\cdot k'_\perp) + \epsilon_{-2s}^i(\epsilon_{2s}\cdot k_\perp)\right]\left[b_{s\underline{k}'}^\dagger b_{s\underline{k}} e^{-i(k'_\perp - k_\perp)\cdot x_\perp} - d_{-s\underline{k}}^\dagger d_{-s\underline{k}'} e^{+i(k'_\perp - k_\perp)\cdot x_\perp}\right], \tag{3.3}$$

where the prime on the sum indicates that terms with $k_\perp = k'_\perp$ are to be excluded [i.e., the global zero mode is removed as per Eq. (2.12)]. We then obtain $\mathring{A}^i$ at $\mathcal{O}(g)$ by inserting (3.3) into Eq. (2.27). Of course, $\mathring{A}^+$ is obtained simply by substituting the expansion of ψ_+ into Eq. (2.16). Neither of these expressions for the zero modes themselves is particularly illuminating, however, and we do not display them.

With the $\mathring{A}^i$ and $\mathring{A}^+$ in hand, we can now construct the Hamiltonian through $\mathcal{O}(g^2)$. For this we need

$$T^{+-} = (\partial_- A^-)^2 + (\partial_+\langle A^+\rangle_0)^2 + \tfrac{1}{2}F^{ij}F^{ij} + g(J^-\langle A^+\rangle_0 - 2J^iA^i) - 4\psi_-^\dagger(i\partial_-\psi_-) + 2\psi_-^\dagger(-i\alpha^i\partial_i + m\beta)\psi_+ + \text{H.c.}, \tag{3.4}$$

where we have discarded terms that will not contribute when integrated. Now the contribution from the normal mode part of the theory may be found in various discussions of QED in DLCQ, for example in Refs. [13,31]. In Appendix B we discuss the contributions from the global zero modes and show they are unnecessary to this order at least. Thus we display here only the parts of P^- arising from the proper zero modes:

$$P_Z^- = \tfrac{1}{2}\int dx^- d^2x_\perp \Big[\tfrac{1}{2}(\partial_i\mathring{A}^j)^2 - \tfrac{1}{2}(\partial_i\mathring{A}^i)^2 + (\partial_+\mathring{A}^+)^2 - 2g\mathring{J}^i\mathring{A}^i + g\mathring{J}^-\mathring{A}^+ - 4\psi_-^{(0)\dagger}(i\partial_-\psi_-^{(2)}) - 4\psi_-^{(2)\dagger}(i\partial_-\psi_-^{(0)})$$
$$+2\psi_-^{(2)\dagger}(-i\alpha^i\partial_i + m\beta)\psi_+ + \text{H.c.}\Big]. \tag{3.5}$$

Here $\psi_-^{(2)}$ is the second-order correction to the dependent Fermi field, which comes entirely from the zero modes:

$$\psi_-^{(2)} = -g\alpha^i\langle A^i\rangle_0\frac{1}{2i\partial_-}\psi_+ + g\langle A^+\rangle_0\frac{1}{(2i\partial_-)^2}(-i\alpha^i\partial_i + m\beta)\psi_+. \tag{3.6}$$

It turns out that all the terms in Eq. (3.5) involving $\psi_-^{(2)}$ cancel among themselves. Furthermore, after implementing the constraints, the terms

$$\tfrac{1}{2}\int dx^- d^2x_\perp\Big[\tfrac{1}{2}(\partial_i\mathring{A}^j)^2 - \tfrac{1}{2}(\partial_i\mathring{A}^i)^2 + (\partial_+\mathring{A}^+)^2 - 2g\mathring{J}^i\mathring{A}^i\Big] \tag{3.7}$$

in Eq. (3.5) may be combined to give

$$\frac{g^2}{2}\int dx^- d^2x_\perp\left[\mathring{J}^i\frac{1}{\partial_\perp^2}\mathring{J}^i\right]_{g=0}. \tag{3.8}$$

Thus the complete contribution to P^- at this order from the proper zero modes reduces to

$$P_Z^- = \frac{g^2}{2}\int dx^- d^2x_\perp\left[\mathring{J}^i\frac{1}{\partial_\perp^2}\mathring{J}^i - \mathring{J}^-\frac{1}{\partial_\perp^2}\mathring{J}^+\right]_{g=0}, \tag{3.9}$$

where we have used Eq. (2.13) to express $\mathring{A}^+$ in terms of $\mathring{J}^+$.

It is now a straightforward (if tedious) exercise to insert the Fourier expansion of ψ_+ into Eq. (3.9) and express P_Z^- in the Fock representation. We obtain new four-fermion operators, as well as fermion bilinears which arise when the four-point terms are brought into normal order (as usual, c-number contributions that result are discarded so that $\langle 0|P^-|0\rangle = 0$). These latter terms have been called "self-induced inertias" in the literature, since they have the Fock space structure of a mass term. One final comment is warranted before we present the explicit terms. The operators $\overset{\circ}{J}^-$ and $\overset{\circ}{J}^+$ in Eq. (3.9) do not commute. Therefore the last term in Eq. (3.9) is non-Hermitian as it stands. This operator-ordering ambiguity is treated by symmetrization:

$$\overset{\circ}{J}^-\frac{1}{\partial_\perp^2}\overset{\circ}{J}^+ \to \frac{1}{2}\left(\overset{\circ}{J}^-\frac{1}{\partial_\perp^2}\overset{\circ}{J}^+ + \overset{\circ}{J}^+\frac{1}{\partial_\perp^2}\overset{\circ}{J}^-\right). \tag{3.10}$$

The results are conveniently grouped into four sets of four-fermion operators and the self-inertias:

$$P_Z^- = P_T^- + P_L^- + P_m^- + P_{m^2}^- + P_{sii}^-, \tag{3.11}$$

where P_T^- and P_L^- are the m-independent contributions from the first and second terms in Eq. (3.9), respectively, P_m^- and $P_{m^2}^-$ are all contributions proportional to m and m^2, respectively, and P_{sii}^- is the full self-inertia contribution. We find

$$\begin{aligned} P_T^- = 2\frac{g^2}{\Omega}\sum_{s,\underline{k},\underline{k}'}{}'\sum_{t,\underline{p},\underline{p}'}{}' &\frac{\delta_{k^+,k'^+}\delta_{p^+,p'^+}}{k^+p^+(k_\perp-k'_\perp)^2(p_\perp-p'_\perp)^2}(k_\perp-k'_\perp)\cdot(p_\perp-p'_\perp)\epsilon^i_{2s}\epsilon^j_{-2s}\left[(k^ip'^j+k'^jp^i)\delta_{s,t}+(k^ip^j+k'^jp'^i)\delta_{s,-t}\right] \\ &\times\left[b^\dagger_{s\underline{k}'}b^\dagger_{t\underline{p}'}b_{s\underline{k}}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}+b^\dagger_{s\underline{k}'}d^\dagger_{-t\underline{p}}b_{s\underline{k}}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}\right. \\ &\left.+d^\dagger_{-s\underline{k}}b^\dagger_{t\underline{p}'}d_{-s\underline{k}'}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}+d^\dagger_{-s\underline{k}}d^\dagger_{-t\underline{p}}d_{-s\underline{k}'}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}\right], \end{aligned} \tag{3.12}$$

$$\begin{aligned} P_L^- = \frac{4g^2}{\Omega}\sum_{s,\underline{k},\underline{k}'}\sum_{t,\underline{p},\underline{p}'}{}' &\frac{\delta_{k^+,k'^+}\delta_{p^+,p'^+}}{(k^+)^2}\frac{\epsilon^i_{2s}\epsilon^j_{-2s}k^ik'^j}{(p_\perp-p'_\perp)^2}\left[b^\dagger_{s\underline{k}'}b^\dagger_{t\underline{p}'}b_{s\underline{k}}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}-b^\dagger_{s\underline{k}'}d^\dagger_{-t\underline{p}}b_{s\underline{k}}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}\right. \\ &\left.-d^\dagger_{-s\underline{k}}b^\dagger_{t\underline{p}'}d_{-s\underline{k}'}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}+d^\dagger_{-s\underline{k}}d^\dagger_{-t\underline{p}}d_{-s\underline{k}'}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}\right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} P_m^- = \frac{2\sqrt{2}g^2m}{\Omega}\sum_{s,\underline{k},\underline{k}'}\sum_{t,\underline{p},\underline{p}'}{}' &\frac{\delta_{k^+,k'^+}\delta_{p^+,p'^+}}{(k^+)^2}\frac{\epsilon^i_{2s}(k^i-k'^i)}{(p_\perp-p'_\perp)^2}\left[b^\dagger_{-s\underline{k}'}b^\dagger_{t\underline{p}'}b_{s\underline{k}}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}-b^\dagger_{-s\underline{k}'}d^\dagger_{-t\underline{p}}b_{s\underline{k}}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}\right. \\ &\left.+d^\dagger_{-s\underline{k}}b^\dagger_{t\underline{p}'}d_{s\underline{k}'}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}-d^\dagger_{-s\underline{k}}d^\dagger_{-t\underline{p}}d_{s\underline{k}'}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}\right], \end{aligned} \tag{3.14}$$

$$\begin{aligned} P_{m^2}^- = \frac{2g^2m^2}{\Omega}\sum_{s,\underline{k},\underline{k}'}\sum_{t,\underline{p},\underline{p}'}{}' &\frac{\delta_{k^+,k'^+}\delta_{p^+,p'^+}}{(k^+)^2}\frac{1}{(p_\perp-p'_\perp)^2}\left[-b^\dagger_{s\underline{k}'}b^\dagger_{t\underline{p}'}b_{s\underline{k}}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p_\perp-p'_\perp}+b^\dagger_{s\underline{k}'}d^\dagger_{-t\underline{p}}b_{s\underline{k}}d_{-t\underline{p}'}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}\right. \\ &\left.+d^\dagger_{-s\underline{k}}b^\dagger_{t\underline{p}'}d_{-s\underline{k}'}b_{t\underline{p}}\delta^{(2)}_{k'_\perp-k_\perp,p'_\perp-p_\perp}-d^\dagger_{s\underline{k}}d^\dagger_{t\underline{p}}d_{s\underline{k}'}d_{t\underline{p}'}\delta_{k'^{(2)}_\perp-k_\perp,p_\perp-p'_\perp}\right], \end{aligned} \tag{3.15}$$

$$P_{sii}^- = \frac{g^2}{\Omega}\sum_{s,\underline{k},\underline{p}}{}'\frac{\delta_{k^+,p^+}}{(k^+)^2(k_\perp-p_\perp)^2}\left[2m^2-(k_\perp^2+p_\perp^2)-2\epsilon^i_{2s}\epsilon^j_{-2s}(k^ip^j+k^jp^i)\right]\left[b^\dagger_{s\underline{p}}b_{s\underline{p}}+d^\dagger_{s\underline{p}}d_{s\underline{p}}\right]. \tag{3.16}$$

For completeness we also display P^+ and P^i, which for $\alpha=0$ have the usual kinematical forms:

$$P^+ = \sum_{s,\underline{p}}p^+\left[b^\dagger_{s\underline{p}}b_{s\underline{p}}+d^\dagger_{s\underline{p}}d_{s\underline{p}}\right]+\sum_{\lambda,\underline{k}}k^+a^\dagger_{\lambda\underline{k}}a_{\lambda\underline{k}}, \tag{3.17}$$

$$P^i = \sum_{s,\underline{p}}p^i\left[b^\dagger_{s\underline{p}}b_{s\underline{p}}+d^\dagger_{s\underline{p}}d_{s\underline{p}}\right]+\sum_{\lambda,\underline{k}}k^ia^\dagger_{\lambda\underline{k}}a_{\lambda\underline{k}}. \tag{3.18}$$

It is straightforward to check that through $\mathcal{O}(g^2)$ all Heisenberg equations reduce to the appropriate field equations and that the Poincaré algebra is correctly obtained. We therefore have a valid representation of the dynamics defined by the equations of motion (2.4)–(2.8).

IV. FERMION SELF-ENERGY

We shall now examine the effects of the zero mode contributions on the one-loop fermion self-energy in this theory. In Ref. [20] it was found that the zero mode contributions to P^- included a counterterm that removed a certain noncovariant, quadratic divergence in the fermion self-energy (eigenvalue of P^-) in a Yukawa theory. We wish to see whether the same thing happens in QED.

The fermion self-energy is not the only quantity to which P_Z^- contributes at $\mathcal{O}(g^2)$, of course. The various four-fermion operators in P_Z^- will certainly contribute to tree-level scattering amplitudes. There will also be divergent contributions to the $e^+e^-\gamma$ vertex, and hence to the charge renormalization, at lowest order. A complete discussion of the effects of the new terms in P^- on the one-loop renormalization of this theory will be presented elsewhere [33].

Let us first discuss the contributions to the fermion self-energy coming from the normal mode sector of the theory. These can essentially be taken from the work of Tang, Brodsky, and Pauli [13,31], with one caveat to be mentioned below. There are two contributions, one coming from one-fermion–one-photon intermediate states (δP_1^-) and one coming from the self-inertia terms in the Hamiltonian (δP_2^-). We find

$$\delta P_1^- = -\frac{\alpha L}{\pi L_\perp^2}\sum_{q_\perp}\sum_{q=2,4,\ldots}^{n-1}\frac{\frac{1}{n(n-q)}\left[n^2(q_\perp-\frac{q}{n}n_\perp)^2+q^2\beta_f\right]+\frac{2n^2}{q^2}(q_\perp-\frac{q}{n}n_\perp)^2}{n^2(q_\perp-\frac{q}{n}n_\perp)^2+q^2\beta_f}, \tag{4.1}$$

$$\delta P_2^- = \frac{\alpha L}{\pi L_\perp^2}\sum_{q_\perp}\left\{\sum_{m=1,3,\ldots}^{\infty}\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{2}\sum_{q=2,4,\ldots}^{\infty}\frac{1}{q(n-q)}+\frac{1}{q(n+q)}\right\}, \tag{4.2}$$

where $(n\pi/L, n_\perp\pi/L_\perp)$ is the momentum of the fermion, $\alpha \equiv \frac{g^2}{4\pi}$, and $\beta_f \equiv (mL_\perp/\pi)^2$. The caveat is that in obtaining Eq. (4.2) we have symmetrized the self-inertias given in Refs. [13,31] under $b \leftrightarrow d$. This is the effect of using the explicitly C-odd form of the current

$$J^\mu = \tfrac{1}{2}[\bar\psi, \gamma^\mu\psi] \tag{4.3}$$

(see, for example, [34]) or, equivalently, of properly symmetrizing products of noncommuting operators in the construction of P^- [35].

Now each of these contributions is separately quadratically divergent in a transverse momentum cutoff. In a continuum formulation, with a suitable (Lorentz covariant) regulator, these quadratic divergences cancel and we recover the expected logarithmic singularity. Here, however, the coefficient of $\frac{\alpha L}{2\pi L_\perp^2}\sum_{q_\perp}$ for $|q_\perp| \to \infty$ is

$$\begin{aligned}\Delta_n \equiv -2\sum_{q=2,4,\ldots}^{n-1}&\left[\frac{1}{n(n-q)}+\frac{2}{q^2}\right]\\ +2\sum_{m=1,3,\ldots}^{\infty}&\left[\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}\right]\\ +\sum_{q=2,4,\ldots}^{\infty}&\left[\frac{1}{q(n-q)}+\frac{1}{q(n+q)}\right].\end{aligned} \tag{4.4}$$

The sums in Eq. (4.4) may be evaluated explicitly to give

$$\Delta_n = \frac{1}{n^2} - \frac{2\ln 2}{n}. \tag{4.5}$$

Thus without the zero modes the fermion self-energy contains a quadratically divergent piece proportional to $\frac{1}{(p^+)^2}$. This would of course need to be removed by a counterterm, but one which cannot correspond simply to the redefinition of a parameter in the Lagrangian. In contrast, the part proportional to $\frac{1}{p^+}$ can be interpreted as a correction to the fermion mass.

Finally, let us consider the contribution from the new self-inertia terms (3.16), which is the sole effect of the zero modes at this order. These give

$$\begin{aligned}\delta P_0^- &= \frac{g^2}{\Omega}\frac{1}{(p^+)^2}\sum_{k_\perp}\frac{1}{(k_\perp-p_\perp)^2}\\ &\quad\times\left[2m^2-(k_\perp^2+p_\perp^2)-2\epsilon_{2s}^i\epsilon_{-2s}^j(k^ip^j+k^jp^i)\right],\end{aligned} \tag{4.6}$$

which is quadratically divergent for $|k_\perp| \to \infty$:

$$\delta P_0^- \approx -\frac{\alpha L}{2\pi L_\perp^2}\frac{1}{n^2}\sum_{k_\perp}. \tag{4.7}$$

The corresponding correction to Δ_n is therefore $-\frac{1}{n^2}$, so that the noncovariant part of the quadratic divergence is in fact canceled when the zero modes are included. The quadratic divergence proportional to $\frac{1}{p^+}$ survives, unlike in the continuum, but this may be removed by a redefinition of the fermion mass. Its occurrence can presumably be traced to the violation of parity that is a generic feature of DLCQ [20].

V. DISCUSSION

We have shown how to perform a general gauge fixing of Abelian gauge theory in DLCQ and cleanly separate the dynamical from the constrained zero-longitudinal-momentum fields. The various zero mode fields *must* be retained in the theory if the equations of motion are to be realized as the Heisenberg equations. We have further seen that taking the constrained fields properly into account renders the ultraviolet behavior of the theory more benign, in that it results in the automatic generation of a counterterm for a noncovariant divergence in the fermion self-energy in lowest-order perturbation theory.

Additional effects of the zero mode contributions to P^-, for example on the charge renormalization, are currently under study [33].

The solutions to the constraint relations for the $\overset{\circ}{A}{}^i$ are all physically equivalent, being related by different choices of gauge in the zero mode sector of the theory. There is a gauge which is particularly simple, however, in that the fields may be taken to satisfy the usual canonical anticommutation relations. This is most easily exposed by examining the kinematical Poincaré generators and finding the solution for which these retain their free-field forms. The unique solution that achieves this is $\varphi = 0$ in Eq. (2.21). For solutions other than this one, complicated commutation relations between the fields will be necessary to correctly translate them in the initial-value surface.

It would be interesting to study the structure of the operators induced by the zero modes from the point of view of the light-cone power-counting analysis of Wilson [26,27]. As noted in the Introduction, to the extent that DLCQ coincides with reality, effects which we would normally associate with the vacuum must be incorporated into the formalism through the new, noncanonical interactions arising from the zero modes. Particularly interesting is the appearance of operators that are nonlocal in the transverse directions [Eq. (3.9)]. These are interesting because the strong infrared effects they presumably mediate could give rise to transverse confinement in the effective Hamiltonian for QCD. There is longitudinal confinement already at the level of the canonical Hamiltonian; that is, effective potential between charges separated only in x^- grows linearly with the separation. This comes about essentially from the nonlocality in x^- (i.e., the small-k^+ divergences) of the light-cone formalism.

It is clearly of interest to develop nonperturbative methods for solving the constraints, since we are ultimately interested in nonperturbative diagonalization of P^-. Several approaches to this problem have recently appeared in the literature [21,23,24] in the context of scalar field theories in 1+1 dimensions. For QED with a realistic value of the electric charge, however, it might be that a perturbative treatment of the constraints could suffice, that is, that we could use a perturbative solution of the constraint to construct the Hamiltonian, which would then be diagonalized nonperturbatively. An approach similar in spirit has been proposed in Ref. [27], where the idea is to use a perturbative realization of the renormalization group to construct an effective Hamiltonian for QCD, which is then solved nonperturbatively. There is some evidence that this kind of approach might be useful. Wivoda and Hiller have recently used DLCQ to study a theory of neutral and interacting charged scalar fields in 3+1 dimensions [36]. They discovered that including four-fermion operators precisely analogous to the perturbative ones appearing in P^-_Z significantly improved the numerical behavior of the simulation.

The extension of the present work to the case of QCD is complicated by the fact that the constraint relations for the gluonic zero modes are nonlinear, as in the ϕ^4 theory. A perturbative solution of the constraints is of course still possible, but in this case, since the effective coupling at the relevant (hadronic) scale is large, it is clearly desirable to go beyond perturbation theory. In addition, because of the central role played by gauge fixing in the present work, we may expect complications due to the Gribov ambiguity [37], which prevents the selection of unique representatives on gauge orbits in nonperturbative treatments of Yang-Mills theory. As a preliminary step in this direction, work is in progress on the pure glue theory in 2+1 dimensions [38]. There it is expected that some of the nonperturbative techniques used recently in 1+1 dimensions [23,24] can be applied.

ACKNOWLEDGMENTS

It is a pleasure to thank G. McCartor, H.-C. Pauli, R. J. Perry, S. S. Pinsky, S. Elser, and F. Wölz for helpful discussions. A.C.K. was supported by the DFG under Contract No. DFG-Gz:Pa 450/1-1 and by NATO under Collaborative Research Grant No. CRG930072, and thanks the Department of Physics of the Ohio State University for its hospitality during a visit where some of this work was conducted. D.G.R. was supported by the National Science Foundation under Grants Nos. PHY-9203145, PHY-9258270, PHY-9207889, and PHY-9102922.

APPENDIX A: NOTATION

1. Light-cone coordinates

We define $x^\pm \equiv x^0 \pm x^3$ and take x^+ to be the evolution parameter. We use Latin indices $(i, j, \ldots)$ to index transverse components $x_\perp = (x^1, x^2)$. A contraction of four-vectors decomposes as $A \cdot B = \frac{1}{2}(A^+B^- + A^-B^+) - A^iB^i$, form which we infer the metric $g_{+-} = g_{-+} = \frac{1}{2}$, $g^{11} = g^{22} = -1$, with all other components vanishing. Derivatives are defined by $\partial_\pm \equiv \partial/\partial x^\pm$, $\partial_i \equiv \partial/\partial x^i$.

We shall also make use of an underscore notation: For position-space variables we write $\underline{x} \equiv (x^-, x_\perp)$, while for momentum-space variables $\underline{k} \equiv (k^+, k_\perp)$. Then $\underline{k} \cdot \underline{x} \equiv \frac{1}{2}k^+x^- - k_\perp \cdot x_\perp$.

We further employ Dirac's notation

$$\alpha^i \equiv \gamma^0\gamma^i, \quad \beta \equiv \gamma^0, \tag{A1}$$

and define $\gamma^\pm \equiv \gamma^0 \pm \gamma^3$. The Hermitian operators

$$\Lambda_\pm \equiv \tfrac{1}{2}\gamma^0\gamma^\pm \tag{A2}$$

serve to project out the constrained and dynamical components of the Fermi field:

$$\psi_\pm \equiv \Lambda_\pm \psi. \tag{A3}$$

In the Dirac representation of the γ matrices,

$$\Lambda_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \tag{A4}$$

which has two eigenvectors, both with eigenvalue 1:

$$\chi_{+1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \chi_{-1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}. \tag{A5}$$

These serve as a convenient spinor basis for the expansion of ψ_+ on the light cone.

2. Field expansions and commutation relations

The mode expansions of the fields on $x^+ = 0$ take the form

$$\psi_+(\underline{x}) = \frac{1}{\sqrt{\Omega}}\sum_{s,\underline{k}} \chi_s \left(b_{s\underline{k}} e^{-i\underline{k}\cdot\underline{x}} + d^\dagger_{-s\underline{k}} e^{i\underline{k}\cdot\underline{x}} \right), \tag{A6}$$

$$A^i_n(\underline{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\lambda,\underline{q}} \epsilon^i_\lambda \left(a_{\lambda\underline{q}} e^{-i\underline{q}\cdot\underline{x}} + a^\dagger_{\lambda\underline{q}} e^{i\underline{q}\cdot\underline{x}} \right), \tag{A7}$$

where $\Omega \equiv 8LL_\perp^2$ is the spatial volume, the spinors χ_s are given in Eq. (A5), and the polarization vectors ϵ^i_λ are defined by

$$\epsilon^i_{+1} = \frac{-1}{\sqrt{2}}(1,i), \quad \epsilon^i_{-1} = \frac{1}{\sqrt{2}}(1,-i). \tag{A8}$$

These satisfy

$$\epsilon^{i*}_\lambda \epsilon^i_{\lambda'} = \delta_{\lambda\lambda'}, \quad \sum_\lambda \epsilon^i_\lambda \epsilon^{j*}_\lambda = \delta^{ij}, \quad \epsilon^{i*}_\lambda = -\epsilon^i_{-\lambda}. \tag{A9}$$

A useful relation satisfied by the χ_s and ϵ^i_λ is

$$\chi^\dagger_{s'}\alpha^i\alpha^j\chi_s = -2\epsilon^i_{-2s}\epsilon^j_{2s}\delta_{s,s'}; \tag{A10}$$

others may be found in Ref. [31]. Recall that the gauge field is taken to be periodic in all coordinates, while the Fermi field is periodic in $x_\perp$ and antiperiodic in x^-. Thus in Eq. (A6) the sum runs over the allowed momenta

$$k^+ = \frac{n\pi}{L}, \quad n = 1,3,5,\ldots,$$

$$k^i = \frac{n^i\pi}{L_\perp}, \quad n^i = 0,\pm1,\pm2,\ldots, \tag{A11}$$

while in (A7) we have

$$q^+ = \frac{m\pi}{L}, \quad m = 2,4,6,\ldots,$$

$$q^i = \frac{m^i\pi}{L_\perp}, \quad m^i = 0,\pm1,\pm2,\ldots\;. \tag{A12}$$

The canonical commutation relations to be imposed are

$$\{\psi_{+\alpha}(\underline{x}), \psi^\dagger_{+\beta}(\underline{x}')\} = (\Lambda_+)_{\alpha\beta}\delta^{(3)}(\underline{x}-\underline{x}'), \tag{A13}$$

$$[A^i_n(\underline{x}), \partial^+ A^j_n(\underline{x}')] = i\delta^{ij}\left[\delta^{(3)}(\underline{x}-\underline{x}') - \frac{1}{2L}\delta^{(2)}(x_\perp - x'_\perp)\right]. \tag{A14}$$

These are realized by the Fock space relations

$$\{b_{s\underline{k}}, b^\dagger_{s'\underline{k}'}\} = \{d_{s\underline{k}}, d^\dagger_{s'\underline{k}'}\} = \delta_{ss'}\delta^{(3)}_{\underline{k},\underline{k}'}, \tag{A15}$$

$$[a_{\lambda\underline{q}}, a^\dagger_{\lambda'\underline{q}'}] = \delta_{\lambda\lambda'}\delta^{(3)}_{\underline{q},\underline{q}'}, \tag{A16}$$

$$\{b,b\} = \{d,d\} = \{b,d^\dagger\} = [a,b] = [a,d] = [a,b^\dagger] = [a,d^\dagger] = 0. \tag{A17}$$

APPENDIX B: THE GLOBAL ZERO MODE SECTOR

As discussed in Ref. [30], the Gauss law in the global zero mode sector is just the vanishing of the total charge:

$$\langle J^+\rangle = 0. \tag{B1}$$

This is a first-class constraint in the Dirac sense [39]. Thus it can only be realized as a condition on physical states in the quantum Hilbert space:

$$:\langle J^+\rangle:|\text{phys}\rangle = 0, \tag{B2}$$

where normal ordering eliminates the zero-point infinity. In terms of the Fock operators,

$$:\langle J^+\rangle := \frac{g}{\Omega}\sum_{s,\underline{k}}(b^\dagger_{s\underline{k}}b_{s\underline{k}} - d^\dagger_{s\underline{k}}d_{s\underline{k}}). \tag{B3}$$

Consequently, to lowest nontrivial order in the coupling constant g, charge singlet states are eigenstates of the global zero modes of the remaining source components:

$$:\langle J^-\rangle := \frac{g}{\Omega}\sum_{s,\underline{k}}\frac{k_\perp^2+m^2}{2k^+}(b^\dagger_{s\underline{k}}b_{s\underline{k}} - d^\dagger_{s\underline{k}}d_{s\underline{k}}), \tag{B4a}$$

$$:\langle J^i\rangle := \frac{g}{\Omega}\sum_{s,\underline{k}}\frac{k^i}{2k^+}(b^\dagger_{s\underline{k}}b_{s\underline{k}} - d^\dagger_{s\underline{k}}d_{s\underline{k}}). \tag{B4b}$$

This is important for the global sector corresponding to A^+. For this we obtain the following contribution to the Hamiltonian:

$$P^-_{\text{glob}} = \Omega\left[\tfrac{1}{2}\langle\pi^-\rangle^2 + g\langle A^+\rangle\langle J^-\rangle\right]. \tag{B5}$$

We observe that $\langle A^+\rangle$ represents a genuine degree of freedom coupling to the electron-photon sector. Thus in lowest-order perturbation theory the Hilbert space can be constructed in terms of the product states

$$\Psi \otimes |\text{phys}\rangle, \tag{B6}$$

where Ψ are stationary wave functions satisfying the Schrödinger equation

$$-\frac{1}{2}\frac{d^2}{dq^2}\Psi + \frac{g}{\Omega}qv\Psi = \mathcal{E}\Psi, \tag{B7}$$

with $q = \Omega\langle A^+\rangle$, $v =: \langle J^-\rangle:$, and $\mathcal{E}$ the energy *density* E/Ω. We cannot solve this exactly, but perturbation theory in the coupling suffices. The free-particle ($g = 0$) wave functions are

$$\Psi^{(0)}(q) = \mathcal{A}e^{i\sqrt{2\mathcal{E}}q}. \tag{B8}$$

There is in fact a boundary condition on Ψ. First, we observe that there is a (rather trivial) Gribov ambiguity in the gauge we employ. Writing the Abelian gauge transformations in the form

$$A^\mu \to UA^\mu U^{-1} + \frac{1}{ig}U\partial^\mu U^{-1}, \tag{B9}$$

we see that $U = \exp(i\frac{2\pi n}{L}x^-)$ maintains the gauge condition $\partial_- A^+ = 0$, as well as the other conditions, but shifts the value of the global zero mode by $2\pi n/gL$:

$$\langle A^+\rangle \to \langle A^+\rangle - \frac{2\pi n}{gL}. \tag{B10}$$

Note that this U is periodic and, due to its x^- dependence, is *not* part of the residual freedom that was left after introducing the condition $\partial_- A^+ = 0$. Thus the quantum mechanical particle q transforms as $q \to q - \frac{2\pi n}{gL}\Omega$. We proceed in analogy with Manton's treatment [28] of the Schwinger model on a circle: The wave function $\Psi^{(0)}$ is assigned the boundary condition

$$\Psi^{(0)}(q) = \Psi^{(0)}\left(q - \frac{2\pi n}{gL}\right). \tag{B11}$$

Now we choose the first "horizon" corresponding to $n = 1$. Within this we obtain the discrete spectrum of "m states," where m is an integer:

$$\Psi_m^{(0)}(q) = \mathcal{A}e^{i\sqrt{2\mathcal{E}_m}q} \quad \text{with} \quad \mathcal{E}_m = \frac{m^2 g^2}{8(2L_\perp)^2}. \tag{B12}$$

Observe that, as expected on dimensional grounds, the longitudinal interval length L has canceled in the discrete values of the energy density. In 1+1 dimensions there is no longer any length parameter left, and so even in the naive continuum limit one obtains a finite energy density. In this case of three space dimensions, the naive $L_\perp \to \infty$ limit collapses all the m states down to zero energy. On the other hand, we now show that for lowest-order perturbation theory, even in the finite volume transitions from any such state to another excitation are suppressed.

The argument rests on the simplicity of the interaction. Since the electron-photon states $|\text{phys}\rangle$ are eigenstates of this Hamiltonian, we can take matrix elements of the Hamiltonian and work with a reduced Schrödinger equation. Evaluating the first-order correction to

$$\langle\Psi_n^{(0)}|\Psi_m^{(0)}\rangle \equiv \int_{-\infty}^{+\infty} dq\Psi_n^{(0)*}(q)\Psi_m^{(0)}(q), \tag{B13}$$

we obtain

$$gv|\mathcal{A}|^2\int_{-\infty}^{+\infty} dq\frac{q}{\mathcal{E}_n - \mathcal{E}_m}e^{i(\sqrt{2\mathcal{E}_n}-\sqrt{2\mathcal{E}_m})q} \quad (n \neq m). \tag{B14}$$

This can be written as the derivative with respect to $\sqrt{2\mathcal{E}_n} - \sqrt{2\mathcal{E}_m}$ of a δ function whose support is *empty*, since $n \neq m$. Thus the correction vanishes, as does the unperturbed amplitude. Thus with the system initialized in a given m state the interactions will not allow a transition to another state. The effect is a pure background that can be ignored as far as the electron-photon theory is concerned. The reader is referred to Ref. [40] for an examination of the role of such states in pure glue theory in 1+1 dimensions.

Let us next discuss $\langle A^i\rangle$. The problem here is that projection of the Maxwell equations into the global sector does not yield information about $\langle A^i\rangle$, neither in the form of a constraint relation nor a dynamical equation of motion. But the Dirac theory sees it reintroduced into the equation for $\overset{\circ}{A^i}$ via the constraint for ψ_-. Physically, these fields represent quanta that propagate along the initial-value surface $x^+ = 0$. It may be, therefore, that they should be initialized along a surface orthogonal to $x^+ = 0$, in a way familiar from, e.g., the treatment of massless fields in two spacetime dimensions. This approach is currently under study. An alternative treatment has been proposed in Ref. [30], in which a mass term is introduced for $\langle A^i\rangle$ only. In this case the global projection of the equation of motion analogous to Eq. (2.8) gives

$$\mu^2\langle A^i\rangle = g\langle J^i\rangle, \tag{B15}$$

so that $\langle A^i\rangle$ becomes constrained. In this approach it is simple to check that the contributions to P^- coming from $\langle A^i\rangle$ begin at $\mathcal{O}(g^3)$ and so would be irrelevant for the present work. Lacking a definite prescription for handling these modes, we shall here simply discard them. It should be noted that (unlike the other zero modes) omitting $\langle A^i\rangle$ from the theory does not introduce inconsistencies into the equations of motion.

Finally, the global zero mode $\langle A^-\rangle$ is set to zero using the only gauge freedom that remains after the conditions of Sec. II are imposed—that of purely x^+-dependent gauge transformations.

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