7-2011

Symmetric Conditions for a Weighted Fourier Transform Inequality

Ryan Berndt
Otterbein University, RBerndt@otterbein.edu

SYMMETRIC CONDITIONS FOR A WEIGHTED
FOURIER TRANSFORM INEQUALITY

Ryan Berndt

ABSTRACT. We discuss conditions on weight functions, necessary or sufficient, so that the Fourier transform is bounded from one weighted Lebesgue space to another. The sufficient condition and the primary necessary condition presented are similar, one being phrased is terms of arbitrary measurable sets and the other in terms of cubes. We believe that the symmetry amongst the two conditions helps frame how a single condition, necessary and sufficient, might appear.

We discuss necessary conditions and a sufficient condition on nonnegative functions u and v such that the following weighted norm inequality holds for the Fourier transform: there exists a constant C such that

$$(1) \qquad \left(\int_{\mathbf{R}^n} |\widehat{f}|^q u\right)^{1/q} \leq C \left(\int_{\mathbf{R}^n} |f|^p v\right)^{1/p}$$

for all $f \in L^p(v)$. Here, we may initially take f to be a simple function and extend the meaning of the Fourier transform and the inequality by density.

A "complete" solution, that is, a single condition on u and v, both necessary and sufficient, is unknown. However, in this note we make some elementary observations concerning a sufficient condition and a necessary condition for (1) to hold. Let $1/p + 1/p' = 1$ and take $1 < p \leq q < \infty$. We say that two sets are *reciprocal* when the product of their measures is one. In Theorem 3 we consider an inequality of the form

$$(2) \qquad \left(\int_A u\right)^{1/q} \left(\int_B v^{-p'/p}\right)^{1/p'} \leq C$$

holding for measurable A and B. If (2) holds for all reciprocal *sets* then the condition is sufficient for (1) to hold. If (2) holds for all reciprocal *cubes* then the condition is necessary. We find the symmetry between the necessary condition and the sufficient condition pleasing, and we believe it helps frame what a complete solution might look like: perhaps condition (2) should hold for some "intermediary" reciprocal sets, somewhere between cubes and measurable sets.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Very little concerning this problem was known before the early 1980's. Notable results can be found in Muckenhoupt [9], Aguilera and Harboure [1], Jurkat and Sampson [8], Benedetto and Heinig [3], Benedetto, Heinig, and Johnson [5, 6], and Sinnamon [10, 11]. A survey of the subject, which includes new proofs and generalizations, can be found in Benedetto and Heinig [4]. Aguilera and Harboure [1] showed, in dimension one, the results contained herein for the cases where $v \equiv 1$ or $u \equiv 1$.

The literature contains inequalities in the form of (2), either as equivalent or in a less general form. We provide the following two theorems as examples. Theorem 1 shows how a less stringent inequality than (2) is enough for decreasing functions, and Theorem 2 shows how an analogous inequality for decreasing rearrangements is enough for (1) to hold more generally.

Theorem 1 [7]. *If $1 < p \leq q < \infty$ and there is a constant C such that*

$$\left(\int_0^{1/s} u \right)^{1/q} \left(\int_0^s v^{-p'/p} \right)^{1/p'} \leq C$$

for all $s > 0$ then

$$\left(\int_0^\infty |\widehat{f}|^q u \right)^{1/q} \leq C \left(\int_0^\infty |f|^p v \right)^{1/p}$$

for all f decreasing.

Theorem 2 [4, p. 7], [5, p. 33], [8]. *Let $1 < p \leq q < \infty$. Let f^* denote the decreasing rearrangement of f. If there is a constant C such that*

$$\left(\int_0^{1/s} u^*(t) \, dt \right)^{1/q} \left(\int_0^s (1/v)^*(t)^{p'/p} \, dt \right)^{1/p'} \leq C$$

for all $s > 0$ then

$$\left(\int_{\mathbf{R}^n} |\widehat{f}|^q u \right)^{1/q} \leq C \left(\int_{\mathbf{R}^n} |f|^p v \right)^{1/p}.$$

Remark. We note that the conditions on the weights in Theorem 1 and Theorem 2 are called $F_{p,q}$ weights and $F_{p,q}^*$ weights in Benedetto, Heinig, and Johnson [5, 6]. With respect to Theorem 1, Sinnamon [10, 11] and Benedetto and Heinig [4, p. 18] have solved an analogous, but different problem, where weighted Lebesgue norms are replaced by weighted Lorentz norms. Also, in Theorem 2, there are known sufficiency results for the case where $p > q$ as well, but we do not use them here.

Before we prove the main result of this note, we set some notation and make some definitions. We define the Fourier transform for $z \in \mathbf{R}^n$ by $\widehat{f}(z) = \int_{\mathbf{R}^n} f(x) e^{-ix \cdot z} \, dx$. For nonnegative functions w, we write $w(E)$ as a shorthand for $\int_E w$. We assume that u and v are nonnegative locally integrable functions (but, if the weighted Fourier inequality holds then $v \in L^1_{loc}$ implies $u \in L^1_{loc}$, cf. Theorem 3). The letter C will denote a constant whose value at each appearance may vary.

Definition. Two measurable sets A and B are reciprocal if $|A||B| = 1$.

Definition. We say that a set of functions F is closed under translations and modulations when $f \in F \iff e^{ix \cdot \tau_1} f(x - \tau_2) \in F$ for all $\tau_1, \tau_2 \in \mathbf{R}^n$.

Definition. We write $(u, v) \in Fourier(p, q)$ for F when there is a constant C such that $\left(\int_{\mathbf{R}^n} |\widehat{f}|^q u \right)^{1/q} \leq C \left(\int_{\mathbf{R}^n} |f|^p v \right)^{1/p}$ for all $f \in F$.

Proposition. *Suppose $F \subset L^1$. If F is closed under translations and modulations, then for all $\tau_1, \tau_2 \in \mathbf{R}^n$*

$$(u, v) \in Fourier(p, q) \text{ for } F \iff$$
$$(u(x + \tau_1), v(x + \tau_2)) \in Fourier(p, q) \text{ for } F.$$

Proof. The proof follows by changing variables and applying the standard identity for the Fourier transform $|(e^{ix \cdot \tau_1} f(x - \tau_2)\widehat{\,}(z)| = |\widehat{f}(z - \tau_1)|$.

Theorem 3. *Let $w = v^{-p'/p}$ and $v \in L^1_{loc}$. In the following statements, (i) $\implies$ (ii) when $1 < p \leq q < \infty$; and (ii) $\implies$ (iii), (iii) $\implies$ (iv) when $1 < p < \infty$ and $q > 0$.*

(i) *There is a constant C such that*

$$u(E)^{1/q} w(F)^{1/p'} \leq C$$

    for all reciprocal sets E and F.

(ii) *There is a constant C such that*

$$\left(\int_{\mathbf{R}^n} |\widehat{f}|^q u \right)^{1/q} \leq C \left(\int_{\mathbf{R}^n} |f|^p v \right)^{1/p}.$$

    for all $f \in L^p(v)$.

(iii) *There is a constant C such that*

$$u(Q)^{1/q} w(P)^{1/p'} \leq C$$

    for all reciprocal cubes Q and P. In particular, u and $w = v^{-p'/p}$ are locally integrable.

(iv) *There is a constant C such that*

$$u(Q)^{1/q} \leq C v(P)^{1/p} |Q|$$

    for all reciprocal cubes Q and P. In particular, if u is nonzero on a set of positive measure then $v \notin L^r$ for any $r < \infty$.

Remark. The conditions in Theorem 3 are not equivalent even in the case of u or v identically equal to 1. That is, (iii) does not imply (i). This fact can be found in Aguilera and Harboure [1, p. 9].

Proof. (i) $\implies$ (ii) The following identity, which can be found in Stein and Weiss [12, p. 202 (3.18)],

$$\int_0^b u^*(t) \, dt = \sup \left\{ \int_E u(x) \, dx : E \subset \mathbf{R}^n, E \text{ measurable}, |E| = b \right\},$$

along with a similar one for w, implies that (i) is equivalent to the hypothesis of Theorem 2. Hence, we are able to conclude (ii).

(ii) $\implies$ (iii) We fix $\varepsilon > 0$ and define the class of functions

$$F = \{\chi_P(x)[v(x-\tau) + \varepsilon]^{-p'/p} e^{i\theta(x)} : P \text{ is cube}, \tau \in \mathbf{R}^n, \theta : \mathbf{R}^n \to \mathbf{R}\}.$$

We note that F is closed under translation and modulations and that $F \subset L^p(v)$, because taking $f \in F$ we have

$$(3) \qquad \int |f|^p v = \int_P \frac{v(x)}{[v(x-\tau) + \varepsilon]^{p'}} \, dx \leq \frac{1}{\varepsilon^{p'}} \int_P v(x) \, dx < \infty,$$

since $v \in L^1_{loc}$. Writing $w_\varepsilon(x) = [v(x) + \varepsilon]^{-p'/p}$ and taking $f = \chi_P w_\varepsilon$ we know by assumption

$$\left(\int_Q |\widehat{\chi_P w_\varepsilon}(z)|^q u(z) \, dz \right)^{1/q} \leq C \left(\int_P w_\varepsilon(x) \, dx \right)^{1/p},$$

for any measurable sets P and Q. We now show (iii) holds for particular cubes Q and P and then apply the translation "invariance" provided by the proposition. The advantage being that we know the behavior of cosine near the origin, but not on arbitrary cubes.

With b an arbitrary positive number we take $P = [0, b]^n$ and $Q = [0, \frac{1}{bn}]^n$. Then, $x \in P$ and $z \in Q$ implies $x \cdot z \in [0, 1]$ and $\cos(x \cdot z) \geq \cos(1)$ so that

$$|\widehat{\chi_P w_\varepsilon}(z)| \geq \left| \int_P w_\varepsilon(x) \cos(x \cdot z) \, dx \right|$$

$$\geq \cos(1) \int_P w_\varepsilon(x) \, dx.$$

Thus,

$$\left(\int_{[0,b]^n} w_\varepsilon(x) \, dx \right) \left(\int_{[0,\frac{1}{bn}]^n} u(z) \, dz \right)^{1/q} \leq C \left(\int_{[0,b]^n} w_\varepsilon(x) \, dx \right)^{1/p}.$$

The integral $w_\varepsilon([0, b]^n)$ is bounded by $\varepsilon^{-p'/p} b^n$ and we may assume not zero (or we would be done), so we can divide both sides by $w_\varepsilon([0, b]^n)^{1/p}$ and then apply the Monotone Convergence Theorem, taking $\varepsilon \to 0$, to get

$$\left(\int_{[0,b]^n} w(x) \, dx \right)^{1/p'} \left(\int_{[0,\frac{1}{bn}]^n} u(z) \, dz \right)^{1/q} \leq C.$$

By the proposition, the above argument holds for u and v replaced with $u(z - \tau_1)$ and $v(x - \tau_2)$ respectively. Hence,

$$(4) \qquad \left(\int_{\tau_2 + [0,b]^n} w(x) \, dx \right)^{1/p'} \left(\int_{\tau_1 + [0,\frac{1}{bn}]^n} u(z) \, dz \right)^{1/q} \leq C.$$

Finally, let P and Q be any cubes such that $|Q| = |P|^{-1}$. We suppose $|P| = b$ and subdivide Q into n^n subcubes Q_i so that $|Q_i| = (bn)^{-n}$. Applying (4), we have

$$u(Q)w(P)^{q/p'} = \sum_{i=1}^{n^n} u(Q_i)w(P)^{q/p'} \le C^q n^n,$$

which proves (iii).

(iii) $\Longrightarrow$ (iv) The inequality in (iv) follows by an application of Hölder's inequality,

$$|P| = \int_P v^{1/p} v^{-1/p} \le v(P)^{1/p} w(P)^{1/p'}.$$

If u is nonzero on a set of positive measure, there is a cube Q_0 such that $u(Q_0) > 0$. Let $d = u(Q_0)^{p/q} C^{-p} |Q_0|^{-p}$. Then, $v(P) \ge d$ for all cubes P with $|P| = |Q_0|^{-1}$, and therefore $v \notin L^r$ for any $r < \infty$.

From the theorem we easily obtain the following corollary, which shows that the Fourier inequality on reciprocal cubes is equivalent to the weighted norm inequality on reciprocal cubes.

Corollary. *Let $w = v^{-p'/p}$. Suppose $1 < p < \infty$ and $q > 0$. There is a constant C such that for all $f \in L^p(v)$ and for all reciprocal cubes Q and P,*

$$\left(\int_Q |\widehat{f}|^q u \right)^{1/q} \le C \left(\int_P |f|^p v \right)^{1/p}$$

if and only if there is a constant C such that

$$u(Q)^{1/q} w(P)^{1/p'} \le C$$

for all reciprocal cubes Q and P.

Proof. We showed the necessity in the previous theorem. The sufficiency is a simple matter of applying the inequality $|\widehat{f}| \le \|f\|_1$ and Hölder's inequality,

$$\int_Q |\widehat{f}(z)|^q u(z)\, dz \le \left(\int_P |f(x)|\, dx \right)^q \int_Q u(z)\, dz$$
$$\le \left(\int_P |f|^p v \right)^{q/p} \left(\int_P v^{-p'/p} \right)^{q/p'} \int_Q u$$
$$\le C^q \left(\int_P |f|^p v \right)^{q/p}.$$

References

1. N. Aguilera and E. Harboure, *On the search for weighted norm inequalities for the Fourier transform*, Pacific J. Math. **104** (1983), 1—14.

2. J. Benedetto and H. Heinig, *Weighted Hardy spaces and the Laplace transform*, Harmonic Analysis Proceedings, Cortona, 1982 (G. Maurceri, F. Ricci, G. Weiss, eds.), Lecture Notes in Mathematics, vol. 992, Springer-Verlag, New York, 1983, pp. 240—277.

3. J. Benedetto and H. Heinig, *Weighted Hardy spaces and the Laplace transform*, Harmonic Analysis, Proceedings, Cortona 1982 (G. Mauceri, F. Ricci, and G. Weiss, eds.), Lecture Notes in Mathematics, vol. 992, Springer-Verlag, New York, 1983, pp. 240—277.

4. J. Benedetto and H. Heinig, *Weighted Fourier inequalities: New Proofs and Generalizations*, J. Fourier Anal. and Appl. **9** (2003), 1—37.

5. J. Benedetto, H. Heinig, and R. Johnson, *Weighted Hardy Spaces and the Laplace Transform II*, Math. Nachr. **132** (1987), 29—55.

6. J. Benedetto, H. Heinig, and R. Johnson, *Fourier Inequalities with A_p-weights*, General Inequalities 5 (W. Walter, ed.), International Conference on General Inequalities, 1986, Oberwolfach, Birkhäuser, Boston, 1987, pp. 217—232.

7. R. Berndt, *A pointwise estimate for the Fourier transform and maxima of a function*, Canad. Math. Bull. (to appear).

8. W. Jurkat and G. Sampson, *On rearrangement and weight inequalities for the Fourier transform*, Indiana Univ. Math. J. **33** (1984), 257—270.

9. B. Muckenhoupt, *Weighted norm inequalities for the Fourier transform*, Trans. Am. Math. Soc. **276** (1983), 729—742.

10. G. Sinnamon, *The Fourier transform in weighted Lorentz spaces*, Publ. Mat. **47** (2003), 3—29.

11. G. Sinnamon, *Fourier inequalities and a new Lorentz space*, Proceedings of the International Symposium on Banach Function Spaces II (ISBFS 2006) (M. Kato and L. Maligranda, eds.), Kyushu Institute of Technology, Kitakyushu, Japan, 14—17, September 2006, Yokohama Publishers, 2008, pp. 145—155.

12. E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, New Jersey, 1971.

Otterbein University, Westerville, Ohio 43081
E-mail address: `rberndt@otterbein.edu`